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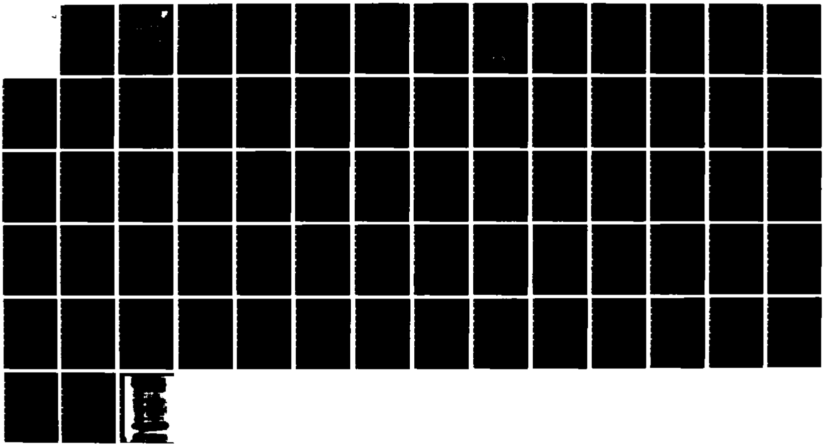

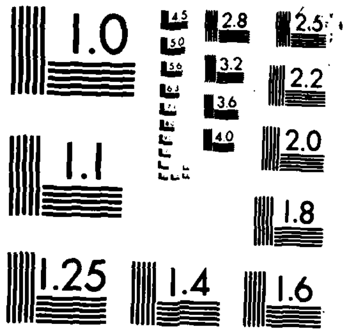

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RADC-TR-84-9, Pt II (of six)
Final Technical Report
April 1984

ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY PERFECTLY CONDUCTING BODIES MOVING IN VACUUM Scattering by Stationary Perfect Conductors

University of Delaware

Allan G. Dallas

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ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, NY 13441

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The equations derived for the problem of electromagnetic scattering from a perfectly conducting body undergoing general non-rigid motion in Part I of this series are specialized for the case when the body is stationary. The simplification of the reformulation of the problem as an integro-differential equation is carried out explicitly in this case and a constructive method of solution is presented.

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ORIENTATION

This is Part II of a six-part report on the results of an investigation into the problem of determining the scattered field resulting from the interaction of a given electromagnetic incident wave with a perfectly conducting body executing specified motion and deformation in vacuum. Part I presents the principal results of the study of the case of a general motion, while Part II contains the specialization and completion of the general reasoning in the situation in which the scattering body is stationary. Part III is devoted to the derivation of a boundary-integral-type representation for the scattered field, in a form involving scalar and vector potentials. Parts IV, V, and VI are of the nature of appendices, containing the proofs of numerous auxiliary technical assertions utilized in the first three parts. Certain of the chapters of Part I are sufficient preparation for studying each of Parts III through VI. Specifically, the entire report is organized as follows:

The section- and equation-numbering scheme is fairly self-explanatory. For example, "[I.5.4]" designates the fourth section of Chapter 5 of Part I, while "(I.5.4.1)" refers to the equation numbered (1) in that section; when the reference is made within Part I, however, these are shortened to "[5.4]" and "(5.4.1)," respectively. Note that Parts II-VI contain no chapter-subdivisions. "[IV.14]" indicates the fourteenth section of Part IV, "(IV.14.6)" the equation numbered (6) within that section; the Roman-numeral designations are never dropped in Parts II-VI.

A more detailed outline of the contents of the entire report appears in [I.1.2]. An index of notations and the bibliography are also to be found in Part I. References to the bibliography are made by citing, for example, "Mikhlin [34]." Finally, it should be pointed out that notations connected with the more common mathematical concepts are standarized for all parts of the report in [I.1.4].

PART II

SCATTERING BY STATIONARY BODIES

We consider in this part the (still non-trivial) scattering of electromagnetic waves by a stationary perfectly conducting body. For certain classes of incident waves and fixed scatterers, we intend to complete the line of reasoning begun in Part I, by producing the solution of the reformulated problem set up in [I.6.1 and 5], and so subsequently generating the solution of the scattering problem.

The first step involves the simplification of the integro-differential equations of [I.6.5] under the assumption of a null motion.

[II.1] THE REFORMULATED PROBLEM IN THE CASE OF A STATIONARY BODY. Suppose, as we shall throughout Part II, that M is a null motion in $\mathbb{M}(2)$: we shall provide the explicit forms of the systems (I.6.5.4 and 6) in this case, by appropriately specializing the results of [I.6.6]. Now, we have $B_\zeta = B_0$ for each $\zeta \in \mathbb{R}$ (so $\mathbb{B} = B_0 \times \mathbb{R}$). The inclusion $M \in \mathbb{M}(2)$ serves merely to ensure that $\partial B_0 = \partial\{B_0^o\}$ is a (2,3;2)-manifold (so B_0^o is a 2-regular domain). To summarize further the simplifications cited in [I.5.13], recall that we have agreed to employ in this stationary case the reference pair (B_0, λ^o)

for $\mathcal{M}$, wherein $X^o(\cdot,0)$ is the identity on $\partial\mathcal{B}_0$: of course, $X^o(\cdot,\zeta)$ is also the identity on $\partial\mathcal{B}_0$ for each $\zeta \in \mathbb{R}$, as is $[X^o]_{(X,t)}$ for each $(X,t) \in \mathbb{R}^4$. Thus,

$$X^o_{,4} = 0 \quad \text{on} \quad \partial\mathcal{B}_0 \times \mathbb{R},$$

and

$$\upsilon = 0 \quad \text{on} \quad \partial\mathcal{B}_0 \times \mathbb{R}.$$

The field ν on $\partial\mathcal{B}_0 \times \mathbb{R}$ is independent of its fourth argument: $\nu(\cdot,\zeta) = \nu(\cdot,0)$ on $\partial\mathcal{B}_0$ for each $\zeta \in \mathbb{R}$. Accordingly,

$$[\nu]_{(X,t)} = \nu(\cdot,0) \quad \text{on} \quad \partial\mathcal{B}_0 \quad \text{for each} \quad (X,t) \in \mathbb{R}^4.$$

Let us write $\nu(\cdot)$ in place of $\nu(\cdot,0)$. It is easy to see that

$$\hat{J}X^o(\cdot,\zeta) = \hat{J}X^o(\cdot,0) := JX^o_0 = 1 \quad \text{for each} \quad \zeta \in \mathbb{R}.$$

If f is an $\mathbb{R}^n$- or $\mathbb{K}$-valued function on $\partial\mathcal{B}_0 \times \mathbb{R}$, there is no distinction between f and $\overset{o}{f}$ with the present choice of reference pair. Moreover, the retardation function τ^o corresponding to $(\mathcal{B}_0, X^o)$ is given by simply

$$\tau^o(Y;X,t) = \frac{1}{c}\, r_X(X^o(Y,t-\tau^o(Y;X,t))) = \frac{1}{c}\, r_X(Y)$$

$$\text{for each} \quad Y \in \partial\mathcal{B}_0 \quad \text{and} \quad (X,t) \in \mathbb{R}^4,$$

whence

$$\tau^o_{;4} = 0,$$

while

$$[f]_{(X,t)}(Y) = f(Y,t-\frac{1}{c}\, r_X(Y)) \quad \text{for} \quad Y \in \partial\mathcal{B}_0 \quad \text{and} \quad (X,t) \in \mathbb{R}^4,$$

if f is defined on $\partial\mathcal{B}_0\times\mathbb{R}$.

Upon taking into account all of these simplifications, and supposing that $\{E^{\prime i},B^{\prime i}\}$ is an incident field appropriate to $\mathcal{M}$ as in [I.4.1], from $(I.6.6.18)_{1,2}$ we infer that the system (I.6.5.4) [(I.6.5.6)] can be given the explicit form, with $\lambda = 1$ $[\lambda = -1]$,

$$\begin{aligned}
\lambda\Psi(Z,\zeta)+ \frac{1}{2\pi}\int_{\partial\mathcal{B}_0} &\frac{1}{r_Z^2}\, r_{Z,k}\nu^k(Z)\cdot[\Psi]_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}\\
&+ \frac{1}{2\pi c}\int_{\partial\mathcal{B}_0} \frac{1}{r_Z}\, r_{Z,k}\nu^k(Z)\cdot[\Psi_{,4}]_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}\\
&+ \frac{1}{2\pi c}\int_{\partial\mathcal{B}_0} \frac{1}{r_Z}\{\nu^k(Z)-\nu^k\}\cdot[\psi^k_{,4}]_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}\\
&= 2\nu^k(Z)\cdot E^{\prime k^c}(Z,\zeta)\\
&[= 2\nu^k(Z)\cdot B^{\prime k}(Z,\zeta)],
\end{aligned} \tag{1$_1$}$$

and

$$\begin{aligned}
\lambda\psi^i(Z,\zeta)+ \frac{1}{2\pi}\int_{\partial\mathcal{B}_0} &\frac{1}{r_Z^2}\, r_{Z,k}\nu^k(Z)\cdot[\psi^i]_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}\\
&+ \frac{1}{2\pi}\int_{\partial\mathcal{B}_0} \frac{1}{r_Z^2}\, r_{Z,i}\{\nu^k-\nu^k(Z)\}\cdot[\psi^k]_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}
\end{aligned}$$

$$+ \frac{1}{2\pi c} \int_{\partial\mathcal{B}_0} \frac{1}{r_Z} r_{Z,k} \nu^k(Z)\cdot[\psi^i_{,4}]_{(Z,\zeta)} \, d\lambda_{\partial\mathcal{B}_0}$$

$$+ \frac{1}{2\pi c} \int_{\partial\mathcal{B}_0} \frac{1}{r_Z} r_{Z,i} \{\nu^k - \nu^k(Z)\}\cdot[\psi^k_{,4}]_{(Z,\zeta)} \, d\lambda_{\partial\mathcal{B}_0} \qquad (1)_2$$

$$= 2\varepsilon_{ijk} \nu^j(Z)\cdot B^{\iota k}(Z,\zeta)$$

$$[= -2\varepsilon_{ijk} \nu^j(Z)\cdot E^{\iota k^c}(Z,\zeta)],$$

$$\text{for each} \quad (Z,\zeta) \in \partial\mathcal{B}_0\times\mathbb{R}.$$

One can also derive the latter equalities by using the expressions given in (I.5.13.5 and 6).

It is to be observed that certain of the troublesome characteristics of the more general equations persist in the systems (1); the retardations of the unknown functions and their 4-derivatives must still be dealt with, although the retardation is independent of the time variable, as are the kernels of the integral operators. We note also that each of the systems (1) is "partially uncoupled." That is, $(1)_2$ involves only the "vector part," ψ, of the unknown; once the solution of this (sub)system has been shown to exist, one can proceed to examine $(1)_1$ for the "scalar part" of the unknown, Ψ.

Again with M a null motion in $\mathbf{M}(2)$ and $\{E^{\iota i}, B^{\iota i}\}$ an incident field appropriate to M, as in [I.4.1], the statements of [I.6.1 and 5] direct us to seek locally Hölder continuous functions Ψ, ψ^i, Γ, and γ^i on $\partial\mathcal{B}_0\times\mathbb{R}$ such that

$$\Psi = \psi^i = \Gamma = \gamma^i = 0 \quad \text{on} \quad \partial B_0 \times (-\infty, 0],$$

$$D_4^j\Psi,\ D_4^j\psi^i,\ D_4^j\Gamma, \text{ and } D_4^j\gamma^i \in C(\partial B_0 \times \mathbb{R}) \quad \text{for} \quad j = 1 \text{ and } 2,$$

and Ψ and ψ^i satisfy (1) with $\lambda = 1$, while Γ and γ^i are solutions of the system (1) with $\lambda = -1$. Recall that, *e.g.*, now

$$[\Psi]_{(Z,\zeta)}(Y) = \Psi(Y, \zeta - r_Z^c(Y)) \quad \text{for} \quad Y,Z \in \partial B_0 \quad \text{and} \quad \zeta \in \mathbb{R}. \tag{2}$$

If we succeed in this, it follows from [I.6.1] and [I.6.5] that there exists a solution of the scattering problem corresponding to M and $\{E^{\iota i}, B^{\iota i}\}$, which can easily be displayed explicitly in terms of either Ψ and ψ^i or Γ and γ^i (cf., [II.9], *infra*). Other relations amongst these functions are cited in [I.6.1]. Now, the systems (1) are similar in form to the single integro-differential equation considered by Fulks and Guenther [17] in the course of carrying out a potential-theoretic investigation of initial-boundary-value problems for the wave equation in a cylindrical domain in $\mathbb{R}^4$. We intend to show here that, under additional hypotheses on B_0 and $\{E^{\iota i}, B^{\iota i}\}$ (corresponding to conditions imposed in [17]), their clever implementation of the familiar technique of successive approximations can be carried over to serve in the examination of (1).

[II.2] S P A C E S   O F   F U N C T I O N S. We begin by establishing notations for the various linear spaces of functions within which

we shall work. We shall have no need to equip these spaces with any sort of locally convex topological structure. Let M be a null motion in $\mathbb{M}(2)$. For $k = 1$ or 3, we define

$$C_4^\infty(\partial B_0\times\mathbb{R};\mathbb{K}^k) := \{\mu:\ \partial B_0\times\mathbb{R}\to\mathbb{K}^k \mid\ D_4^j\mu \in C(\partial B_0\times\mathbb{R};\mathbb{K}^k) \text{ for each } j\in\mathbb{N}\}, \tag{1}$$

$$\begin{aligned}\mathcal{E}_{4,0}(\partial B_0\times\mathbb{R};\mathbb{K}^k) := \{\mu\in C_4^\infty(\partial B_0\times\mathbb{R};\mathbb{K}^k) \mid\ & \mu = 0 \text{ on } \partial B_0\times(-\infty,0];\\ & \text{for each } T>0, \text{ there exist } b_{\mu,T}>0,\\ & c_{\mu,T}>0, \text{ and } \delta_{\mu,T}\in(0,1) \text{ such that}\\ & |\mu(Z,\zeta)|_k \le b_{\mu,T},\\ & |D_4^j\mu(Z,\zeta)|_k \le b_{\mu,T}c_{\mu,T}^j\cdot j^{(1+\delta_{\mu,T})j}\\ & \text{for } Z\in\partial B_0,\ 0<\zeta\le T, \text{ and } j\in\mathbb{N}\},\end{aligned} \tag{2}$$

and

$$\mathcal{E}^H_{4,0}(\partial B_0\times\mathbb{R};\mathbb{K}^k) := \{\mu\in\mathcal{E}_{4,0}(\partial B_0\times\mathbb{R};\mathbb{K}^k) \mid\ \mu \text{ is locally Hölder continuous}\}. \tag{3}$$

We shall write simply $C_4^\infty(\partial B_0\times\mathbb{R})$ in place of $C_4^\infty(\partial B_0\times\mathbb{R};\mathbb{K})$, etc. The utility of the estimates imposed on the 4-derivatives of an element μ of either $\mathcal{E}_{4,0}(\partial B_0\times\mathbb{R})$ or $\mathcal{E}_{4,0}(\partial B_0\times\mathbb{R};\mathbb{K}^3)$ will become apparent in [II.7]. In [17], it is pointed out that $\mathcal{E}_{4,0}(\partial B_0\times\mathbb{R})$ is large enough to be dense in "most standard functions spaces" on

$\partial\mathcal{B}_0\times[0,\infty)$; this can be verified by constructing mollified functions to approximate a given function, in which the mollifying kernel is chosen to lie in $\mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$. Fulks and Guenther also note that if one were to allow $\delta_{\mu,T} = 0$ in the definition (2), then it would follow that $\mu(Z,\cdot)$ is analytic for each $Z \in \partial\mathcal{B}_0$, so $\mu = 0$, since it vanishes on $\partial\mathcal{B}_0\times(-\infty,0]$.

[II.3] O P E R A T O R S. It is also convenient to introduce concise notations for the operators figuring in the integro-differential equations which we are to study. For the null motion $M \in \mathbb{M}(2)$, we employ the usual reference pair $(\mathcal{B}_0, X^o)$ and the modified notation $\nu(\cdot)$ for $\nu(\cdot,\zeta)$ $(\zeta \in \mathbb{R})$; cf., [II.1].

We find it necessary to begin by citing facts concerning certain auxiliary functions on $\partial\mathcal{B}_0\times\mathbb{R}$, following from the general considerations of Part IV. Let $(Y,Z,\zeta) \mapsto \phi_{(Z,\zeta)}(Y)$ be a continuous function on $\partial\mathcal{B}_0\times\partial\mathcal{B}_0\times\mathbb{R}$. Noting that $\mathcal{B}_0^o$ is a Lyapunov domain, we can define $\mathcal{W}_1^*\{\phi\}$ on $\partial\mathcal{B}_0\times\mathbb{R}$ according to

$$\mathcal{W}_1^*\{\phi\}(Z,\zeta) := \frac{1}{4\pi}\int_{\partial\mathcal{B}_0} \frac{1}{r_Z^2}\, r_{Z,k}\nu^k\cdot\phi_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}, \tag{1}$$

$$\text{for each} \quad (Z,\zeta) \in \partial\mathcal{B}_0\times\mathbb{R};$$

this is just the specialization of Definition [IV.20] to the present case of a null motion. Next, suppose that $(Y,Z) \mapsto \Gamma_Z(Y)$ is bounded and continuous on the set $\{(Y,Z) \mid Y \in \partial\mathcal{B}_0,\ Z \in \partial\mathcal{B}_0,\ Y \neq Z\}$: then the function $\mathcal{W}_{31}^*\{\phi\}$ is given on $\partial\mathcal{B}_0\times\mathbb{R}$ by

$$\mathcal{W}^*_{31}\{\phi\}(Z,\zeta) := \int_{\partial\mathcal{B}_0} \frac{1}{r_Z} \cdot \Gamma_Z \cdot \phi_{(Z,\zeta)} \, d\lambda_{\partial\mathcal{B}_0}, \tag{2}$$

$$\text{for each} \quad (Z,\zeta) \in \partial\mathcal{B}_0 \times \mathbb{R};$$

the existence of the integral here follows from the considerations of Definition [IV.30.i]. Finally, assume that ϕ is also such that

$$\left.\begin{array}{l} \text{whenever } K \subset \mathbb{R} \text{ is compact, there exist } \ell_K > 0, \\ \Delta_K > 0, \text{ and } \alpha_K \in (0,1] \text{ for which} \\ |\phi_{(Z,\zeta)}(Y)| \leq \ell_K \cdot r_Z^{\alpha_K}(Y) \quad \text{for} \quad Z \in \partial\mathcal{B}_0, \quad \zeta \in K, \\ \qquad \text{and} \quad Y \in \partial\mathcal{B}_0 \cap B^3_{\Delta_K}(Z). \end{array}\right\} \tag{3}$$

Under this hypothesis, it is easy to see that we can define $\mathcal{W}^*_{32}\{\phi\}$ on $\partial\mathcal{B}_0 \times \mathbb{R}$ by

$$\mathcal{W}^*_{32}\{\phi\}(Z,\zeta) := \int_{\partial\mathcal{B}_0} \frac{1}{r_Z^2} \cdot \Gamma_Z \cdot \phi_{(Z,\zeta)} \, d\lambda_{\partial\mathcal{B}_0}, \tag{4}$$

$$\text{for each} \quad (Z,\zeta) \in \partial\mathcal{B}_0 \times \mathbb{R};$$

cf., Definition [IV.30.ii].

Now, consider the following hypothesis on the function $(Y,Z,\zeta) \mapsto \phi_{(Z,\zeta)}(Y)$, continuous on $\partial\mathcal{B}_0 \times \partial\mathcal{B}_0 \times \mathbb{R}$:

$$\left.\begin{array}{l}\text{for each compact subset } K \subset \mathbb{R}, \text{ there can be} \\ \text{found } \kappa_K > 0 \text{ and } \beta_K \in (0,1] \text{ such that} \\ |\phi_{(Z_2,\zeta_2)}(Y)-\phi_{(Z_1,\zeta_1)}(Y)| \leq \kappa_K \cdot |(Z_2,\zeta_2)-(Z_1,\zeta_1)|_4^{\beta_K} \\ \text{whenever } Y \in \partial B_0, \quad Z_1,Z_2 \in \partial B_0, \\ \text{and } \zeta_1,\zeta_2 \in K.\end{array}\right\} \tag{5}$$

If (5) holds, then $W_1^*\{\phi\}$ is locally Hölder continuous on $\partial B_0 \times \mathbb{R}$, *i.e.*, is Hölder continuous on each compact subset of $\partial B_0 \times \mathbb{R}$; this follows directly from [IV.24]. If the bounded and continuous function $(Y,Z) \mapsto \Gamma_Z(Y)$ on $\{(Y,Z) \mid Y \in \partial B_0, Z \in \partial B_0, Y \neq Z\}$ satisfies the condition

$$\left.\begin{array}{l}\text{there exist } \kappa_1 > 0, \ \kappa_2 > 0, \ \Delta_0 > 0, \text{ and} \\ \beta_0 \in (0,1] \text{ such that} \\ |\Gamma_{Z_2}(Y)-\Gamma_{Z_1}(Y)| \leq \kappa_1 \cdot |Z_2-Z_1|_3^{\beta_0} + \dfrac{\kappa_2}{r_{Z_1}(Y)} \cdot |Z_2-Z_1|_3 \\ \text{for } Z_1,Z_2 \in \partial B_0 \text{ with } |Z_2-Z_1|_3 \leq \Delta_0, \\ \text{and } Y \in \partial B_0 \cap \{Z_1\}' \cap \{Z_2\}'\end{array}\right\} \tag{6}$$

while (5) holds for ϕ, then $W_{31}^*\{\phi\}$ is locally Hölder continuous on $\partial B_0 \times \mathbb{R}$; if, in addition, ϕ fulfills (3), then $W_{32}^*\{\phi\}$ is also locally Hölder continuous on $\partial B_0 \times \mathbb{R}$. These assertions concerning $W_{31}^*\{\phi\}$ and $W_{32}^*\{\phi\}$ are consequences of conclusions (ii)' and (iii)' of Theorem [IV.31], respectively.

Again supposing that $(Y,Z,\zeta) \mapsto \phi_{(Z,\zeta)}(Y)$ is continuous on

$\partial\mathcal{B}_0\times\partial\mathcal{B}_0\times\mathbb{R}$ and (5) holds, if we now introduce functions $\tilde{\mathcal{W}}_{31}\{\phi\}$, $\tilde{\mathcal{W}}_{31ij}\{\phi\}$, and $\hat{\mathcal{W}}_{31i}\{\phi\}$ on $\partial\mathcal{B}_0\times\mathbb{R}$ according to

$$\tilde{\mathcal{W}}_{31}\{\phi\}(Z,\zeta) := \int_{\partial\mathcal{B}_0} \frac{1}{r_Z}\cdot r_{Z,k}\nu^k\cdot\phi_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}, \tag{7}$$

$$\tilde{\mathcal{W}}_{31ij}\{\phi\}(Z,\zeta) := \int_{\partial\mathcal{B}_0} \frac{1}{r_Z}\, r_{Z,i}\{\nu^j-\nu^j(Z)\}\cdot\phi_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0}, \tag{8}$$

and

$$\hat{\mathcal{W}}_{31i}\{\phi\}(Z,\zeta) := \int_{\partial\mathcal{B}_0} \frac{1}{r_Z}\{\nu^i-\nu^i(Z)\}\cdot\phi_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0} \tag{9}$$

$$\text{for}\quad (Z,\zeta)\in\partial\mathcal{B}_0\times\mathbb{R},$$

it is obvious we can use the cited property of $\mathcal{W}^*_{31}\{\phi\}$ three times in order to deduce that each of these is locally Hölder continuous on $\partial\mathcal{B}_0\times\mathbb{R}$. For example, in the case of $\tilde{\mathcal{W}}_{31}\{\phi\}$, we can take $\Gamma_Z(Y)$ as $r_{Z,k}(Y)\cdot\nu^k(Y)$ and show that (6) is fulfilled thereby, noting that

$$\begin{aligned}|r_{Z_2,k}(Y)\cdot\nu^k(Y)-r_{Z_1,k}(Y)\cdot\nu^k(Y)| &\le |\text{grad } r_{Z_2}(Y) - \text{grad } r_{Z_1}(Y)| \\ &\le \frac{2}{r_{Z_1}(Y)}\cdot|Z_2-Z_1|_3.\end{aligned}$$

Next, maintaining the hypotheses on ϕ, let us set

$$\tilde{\varphi}_{(Z,\zeta)}(Y) := \{\nu^j(Y)-\nu^j(Z)\}\cdot\phi_{(Z,\zeta)}(Y)$$

$$\text{for}\quad Y,Z\in\partial\mathcal{B}_0 \quad\text{and}\quad \zeta\in\mathbb{R},$$

and take $\Gamma_Z(Y)$ to be given by $r_{Z,i}(Y)$ (so that (6) is true). Observing that ν is Lipschitz continuous on $\partial\mathcal{B}_0$ in the present setting, whenever K is compact in $\mathbb{R}$ we have

$$|\tilde{\phi}_{(Z,\zeta)}(Y)| \leq \{\max_{\substack{\hat{Y},\hat{Z} \in \partial\mathcal{B}_0 \\ \hat{\zeta} \in K}} \tilde{\phi}_{(\hat{Z},\hat{\zeta})}(\hat{Y})\}\cdot a\cdot r_Y(Z)$$

$$\text{for} \quad Y,Z \in \partial\mathcal{B}_0 \quad \text{and} \quad \zeta \in K,$$

for some $a > 0$, so that (3) holds when ϕ is replaced therein by $\tilde{\phi}$. Further, since ϕ satisfies (5), it is easy to check that $\tilde{\phi}$ also fulfills a condition of the form of (5). In consequence of these facts, we can define $\mathcal{W}^*_{1ij}\{\phi\}$ on $\partial\mathcal{B}_0\times\mathbb{R}$, as a function of the form $\mathcal{W}^*_{32}\{\tilde{\phi}\}$, by setting

$$\mathcal{W}^*_{1ij}\{\phi\}(Z,\zeta) := \frac{1}{4\pi}\int_{\partial\mathcal{B}_0} \frac{1}{r_Z^2}\, r_{Z,i}\{\nu^j-\nu^j(Z)\}\cdot\phi_{(Z,\zeta)}\, d\lambda_{\partial\mathcal{B}_0},$$

$$\text{for} \quad (Z,\zeta) \in \partial\mathcal{B}_0\times\mathbb{R}, \tag{10}$$

and assert that $\mathcal{W}^*_{1ij}\{\phi\}$ is locally Hölder continuous on $\partial\mathcal{B}_0\times\mathbb{R}$.

Turning next to the definitions of the operators in which we are primarily interested, we first suppose that $\mu \in C(\partial\mathcal{B}_0\times\mathbb{R})$ is such that $\mu_{,4} \in C(\partial\mathcal{B}_0\times\mathbb{R})$, and define the corresponding function $L\mu\colon\ \partial\mathcal{B}_0\times\mathbb{R} \to \mathbb{K}$ according to

$$L\mu(Z,\zeta) := -\frac{1}{2\pi}\int_{\partial\mathcal{B}_0} L_Z\cdot\{[\mu]_{(Z,\zeta)}+r_Z^c[\mu_{,4}]_{(Z,\zeta)}\}\, d\lambda_{\partial\mathcal{B}_0},$$

$$\text{for} \quad Z \in \partial\mathcal{B}_0, \quad \zeta \in \mathbb{R}, \tag{11}$$

wherein

$$L_Z(Y) := \frac{1}{r_Z^2(Y)} r_{Z,k}(Y)\nu^k(Z) \quad \text{for} \quad Z \in \partial\mathcal{B}_0, \quad Y \in \partial\mathcal{B}_0 \cap \{Z\}'; \qquad (12)$$

clearly,

$$L\mu = -2\mathcal{W}_1^*\{[\mu]\}+2\mathcal{W}_{1kk}^*\{[\mu]\} - \frac{1}{2\pi c}\tilde{\mathcal{W}}_{31}\{[\mu,_4]\} + \frac{1}{2\pi c}\tilde{\mathcal{W}}_{31kk}\{[\mu,_4]\}. \qquad (13)$$

If $\tilde{\mu} \in C(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$ with $\tilde{\mu},_4 \in C(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$, we take the function $\mathbb{L}\tilde{\mu}$: $\partial\mathcal{B}_0\times\mathbb{R} \to \mathbb{K}^3$ to be given by

$$\{\mathbb{L}\tilde{\mu}\}^i(Z,\zeta) := -\frac{1}{2\pi}\int_{\partial\mathcal{B}_0} L_Z^{ij}\{[\tilde{\mu}^j]_{(Z,\zeta)}+r_Z^c[\tilde{\mu}^j,_4]_{(Z,\zeta)}\}\, d\lambda_{\partial\mathcal{B}_0}, \qquad (14)$$

$$\text{for} \quad Z \in \partial\mathcal{B}_0, \quad \zeta \in \mathbb{R},$$

wherein

$$L_Z^{ij}(Y) := \frac{1}{r_Z^2(Y)}\{r_{Z,i}(Y)\nu^j(Y)-\varepsilon_{ik\ell}\varepsilon_{\ell mj}r_{Z,m}(Y)\nu^k(Z)\}$$

$$= \frac{1}{r_Z^2(Y)}\{r_{Z,k}(Y)\nu^k(Z)\delta^{ij}+r_{Z,i}(Y)\{\nu^j(Y)-\nu^j(Z)\}\}, \qquad (15)$$

$$\text{for} \quad Z \in \partial\mathcal{B}_0, \quad Y \in \partial\mathcal{B}_0\cap\{Z\}';$$

one can easily check that

$$\{\mathbb{L}\tilde{\mu}\}^i = -2\mathcal{W}_1^*\{[\tilde{\mu}^i]\}+2\mathcal{W}_{1kk}^*\{[\tilde{\mu}^i]\}-2\mathcal{W}_{1ik}^*\{[\tilde{\mu}^k]\}$$
$$- \frac{1}{2\pi c}\tilde{\mathcal{W}}_{31}\{[\tilde{\mu}^i,_4]\}+\frac{1}{2\pi c}\tilde{\mathcal{W}}_{31kk}\{[\tilde{\mu}^i,_4]\}-\frac{1}{2\pi c}\tilde{\mathcal{W}}_{31ik}\{[\tilde{\mu}^k,_4]\}. \qquad (16)$$

Finally, with $\tilde{\mu}$ as in the preceding definition, the function $\Lambda\tilde{\mu}$: $\partial\mathcal{B}_0\times\mathbb{R} \to \mathbb{K}$ is given by

$$\Lambda\tilde{\mu}(Z,\zeta) := \frac{1}{2\pi c} \int_{\partial B_0} \frac{1}{r_Z} \{\nu^k - \nu^k(Z)\}\cdot[\tilde{\mu}^k_{,4}]_{(Z,\zeta)}\, d\lambda_{\partial B_0}, \tag{17}$$

$$\text{for} \quad Z \in \partial B_0, \quad \zeta \in \mathbb{R},$$

so that

$$\Lambda\tilde{\mu} = \frac{1}{2\pi c}\,\hat{W}_{31k}\{[\tilde{\mu}^k_{,4}]\}. \tag{18}$$

Let us make several observations concerning the operators so defined.

(i) Suppose that μ, $\mu_{,4}$, and $\mu_{,44}$ lie in $C(\partial B_0 \times \mathbb{R})$. Let K be a compact subset of $\mathbb{R}$. If $Y \in \partial B_0$, Z_1, $Z_2 \in \partial B_0$, and ζ_1, $\zeta_2 \in K$, then, supposing without loss that $\zeta_1 \leq \zeta_2$, we may apply the mean-value theorem to write, for some $\zeta_{12}(Y)$ lying between $\zeta_1 - r^c_{Z_1}(Y)$ and $\zeta_2 - r^c_{Z_2}(Y)$ or equal to their common value if these numbers are not distinct,

$$\begin{aligned}
&|[\mu]_{(Z_2,\zeta_2)}(Y) - [\mu]_{(Z_1,\zeta_1)}(Y)| \\
&= |\mu(Y,\zeta_2 - r^c_{Z_2}(Y)) - \mu(Y,\zeta_1 - r^c_{Z_1}(Y))| \\
&= |\mu_{,4}(Y,\zeta_{12}(Y))|\cdot|\{\zeta_2 - r^c_{Z_2}(Y)\} - \{\zeta_1 - r^c_{Z_1}(Y)\}| \\
&\leq |\mu_{,4}(Y,\zeta_{12}(Y))|\cdot\{|\zeta_2 - \zeta_1| + \frac{1}{c}\,|Z_2 - Z_1|_3\} \\
&\leq \left\{ \sup_{\substack{Z \in \partial B_0 \\ \zeta_1 - \frac{1}{c}\operatorname{diam} B_0 \leq \zeta \leq \zeta_2}} |\mu_{,4}(Z,\zeta)| \right\}\cdot\left(1 + \frac{1}{c^2}\right)^{1/2}\cdot|(Z_2,\zeta_2) - (Z_1,\zeta_1)|_4 .
\end{aligned}$$

We can derive a corresponding estimate with $\mu_{,4}$ replacing μ. Thus, we have shown that $[\mu]$ and $[\mu_{,4}]$ satisfy hypothesis (5), so that equality (13) and the remarks made concerning the local

Hölder continuity of $\mathcal{W}^*_1\{\phi\}$, $\mathcal{W}^*_{1ij}\{\phi\}$, $\tilde{\mathcal{W}}_{31}\{\phi\}$, and $\tilde{\mathcal{W}}_{31ij}\{\phi\}$, for ϕ satisfying (5), allow us to conclude that $L\mu$ is locally Hölder continuous when μ possesses the properties required. For example, if $\mu \in C^\infty_4(\partial\mathcal{B}_0\times\mathbb{R})$, then $L\mu$ is locally Hölder continuous on $\partial\mathcal{B}_0\times\mathbb{R}$.

Reasoning similarly, we can deduce that $\mathbb{L}\tilde{\mu}$ and $\Lambda\tilde{\mu}$ are locally Hölder continuous if $\tilde{\mu}$, $\tilde{\mu}_{,4}$, and $\tilde{\mu}_{,44}$ are elements of $C(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$ (which is true if, say, $\tilde{\mu} \in C^\infty_4(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$); here, of course, we appeal to (16) and (18).

(ii) Again, let $\mu \in C(\partial\mathcal{B}_0\times\mathbb{R})$ be such that $\mu_{,4}$ and $\mu_{,44}$ are also in $C(\partial\mathcal{B}_0\times\mathbb{R})$. In view of the definition (11), and keeping in mind (II.1.2), it is an easy exercise to show that $D_4L\mu$ exists on $\partial\mathcal{B}_0\times\mathbb{R}$ (cf., Lemma [IV.7]), and

$$D_4L\mu = LD_4\mu \tag{19}$$

(and that $(D_4L\mu)(Z,\cdot) \in C(\mathbb{R})$ for each $Z \in \partial\mathcal{B}_0$). If, in addition, it is known that $\mu_{,444} \in C(\partial\mathcal{B}_0\times\mathbb{R})$, then (i) and (19) imply that $D_4L\mu$ is locally Hölder continuous. Consequently, it is clear that for $\mu \in C^\infty_4(\partial\mathcal{B}_0\times\mathbb{R})$, we also have $L\mu \in C^\infty_4(\partial\mathcal{B}_0\times\mathbb{R})$, with $D^j_4L\mu$ being locally Hölder continuous and

$$D^j_4L\mu = LD^j_4\mu, \qquad \text{for each} \quad j \in \mathbb{N}. \tag{20}$$

In an analogous fashion, for $\tilde{\mu} \in C^\infty_4(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$, one can show that $D^j_4\mathbb{L}\tilde{\mu}$ and $D^j_4\Lambda\tilde{\mu}$ exist and are locally Hölder continuous on $\partial\mathcal{B}_0\times\mathbb{R}$, with

$$\left.\begin{aligned} D_4^j \mathbb{L}\tilde{\mu} &= \mathbb{L}D_4^j\tilde{\mu}, \quad (21) \\ \text{and} \qquad D_4^j \Lambda\tilde{\mu} &= \Lambda D_4^j\tilde{\mu}, \quad (22) \end{aligned}\right\} \text{ for each } \quad j \in \mathbb{N}:$$

in particular, $\mathbb{L}\tilde{\mu} \in C_4^\infty(\partial B_0 \times \mathbb{R};\mathbb{K}^3)$, and $\Lambda\tilde{\mu} \in C_4^\infty(\partial B_0 \times \mathbb{R})$.

(iii) We shall have need of various facts pertaining to the iterates of the operators L and $\mathbb{L}$. First, since L maps $C_4^\infty(\partial B_0 \times \mathbb{R})$ into itself, it is evident that the sequence of iterated operators $\{L^n\}_{n=0}^\infty$ is well-defined in this linear space, wherein

$$\left.\begin{aligned} L^o\mu &:= \mu, \\ L^n\mu &:= LL^{n-1}\mu \quad \text{for each} \quad n \in \mathbb{N}, \end{aligned}\right\} \text{ for each } \quad \mu \in C_4^\infty(\partial B_0 \times \mathbb{R}). \quad (23)$$

Suppose that $\mu \in C_4^\infty(\partial B_0 \times \mathbb{R})$: explicitly, from (11) we have

$$\begin{aligned} L^2\mu(Z_0,t) &:= \{L(L\mu)\}(Z_0,t) \\ &= -\frac{1}{2\pi} \int_{\partial B_0} L_{Z_0}(Z_1)\cdot\{(1+r^c_{Z_0}(Z_1)D_4)L\mu\}(Z_1,t-r^c_{Z_0}(Z_1))\, d\lambda_{\partial B_0}(Z_1) \\ &= -\frac{1}{2\pi} \int_{\partial B_0} L_{Z_0}(Z_1)\cdot\{L((1+r^c_{Z_0}(Z_1)D_4)\mu)\}(Z_1,t-r^c_{Z_0}(Z_1))\, d\lambda_{\partial B_0}(Z_1) \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{\partial B_0} \int_{\partial B_0} L_{Z_0}(Z_1)L_{Z_1}(Z_2) \\ &\qquad \cdot\{(1+r^c_{Z_1}(Z_2)D_4)(1+r^c_{Z_0}(Z_1)D_4)\mu\}(Z_2,t-r^c_{Z_0}(Z_1)-r^c_{Z_1}(Z_2)) \\ &\qquad\qquad d\lambda_{\partial B_0}(Z_2)\, d\lambda_{\partial B_0}(Z_1); \end{aligned}$$

in fact, by induction it is not hard to prove that

$$L^n\mu(Z_0,t) = \left(\frac{-1}{2\pi}\right)^n \int_{\partial\mathcal{B}_0} \cdots \int_{\partial\mathcal{B}_0} \left\{\prod_{i=0}^{n-1} L_{Z_i}(Z_{i+1})\right\}$$

$$\cdot\left\{\left[\prod_{j=0}^{n-1} \{1+r^c_{Z_j}(Z_{j+1})D_4\}\right]\mu\right\}\left(Z_n, t-\sum_{k=0}^{n-1} r^c_{Z_k}(Z_{k+1})\right)$$

$$d\lambda_{\partial\mathcal{B}_0}(Z_n)\ \ldots\ d\lambda_{\partial\mathcal{B}_0}(Z_1), \tag{24}$$

for $Z_0 \in \partial\mathcal{B}_0$, $t \in \mathbb{R}$, and $n \in \mathbb{N}$.

Moreover, using induction on n, we can show that, for each $\mu \in C_4^\infty(\partial\mathcal{B}_0\times\mathbb{R})$, $n \in \mathbb{N}$, and $j \in \mathbb{N}$, $D_4^jL^n\mu$ exists and is locally Hölder continuous on $\partial\mathcal{B}_0\times\mathbb{R}$, with

$$D_4^jL^n\mu = L^nD_4^j\mu. \tag{25}$$

For, we have already seen this to be so if $n = 1$, in (i) and (ii). If $\tilde{n} \in \mathbb{N}$ and the result is assumed true for $\tilde{n}$ and for each $j \in \mathbb{N}$, then (i) and (ii) can again be applied, since $L^{\tilde{n}}\mu \in C_4^\infty(\partial\mathcal{B}_0\times\mathbb{R})$, giving, for each $j \in \mathbb{N}$,

$$D_4^jL^{\tilde{n}+1}\mu = D_4^jLL^{\tilde{n}}\mu = LD_4^jL^{\tilde{n}}\mu = LL^{\tilde{n}}D_4^j\mu = L^{\tilde{n}+1}D_4^j\mu,$$

and implying the local Hölder continuity of $D_4^jL^{\tilde{n}+1}\mu$, because of the equality $D_4^jL^{\tilde{n}+1}\mu = LD_4^jL^{\tilde{n}}\mu$.

Since $\mathbb{L}$ takes $C_4^\infty(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$ into itself, the iterates of this operator can be defined by

$$\left.\begin{array}{ll} \mathbb{L}^{0}\tilde{\mu} := \tilde{\mu}, & \\ \mathbb{L}^{n}\tilde{\mu} := \mathbb{L}\mathbb{L}^{n-1}\tilde{\mu} & \text{for each} \quad n \in \mathbb{N}, \end{array}\right\} \text{ for each } \quad \tilde{\mu} \in C_4^{\infty}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3). \tag{26}$$

Explicitly, for each $\tilde{\mu} \in C_4^{\infty}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$, we find

$$\{\mathbb{L}^{n}\tilde{\mu}\}^{i_0}(Z_0,t) = \left(\frac{-1}{2\pi}\right)^{n} \int_{\partial\mathcal{B}_0} \cdots \int_{\partial\mathcal{B}_0} \left\{\prod_{\ell=0}^{n-1} L_{Z_\ell}^{i_\ell i_{\ell+1}}(Z_{\ell+1})\right\}$$
$$\cdot\left\{\left(\prod_{j=0}^{n-1} \{1+r^{c}_{Z_j}(Z_{j+1})D_4\}\right)\tilde{\mu}^{i_n}\right\}(Z_n, t-\sum_{k=0}^{n-1} r^{c}_{Z_k}(Z_{k+1})) \tag{27}$$
$$d\lambda_{\partial\mathcal{B}_0}(Z_n) \cdots d\lambda_{\partial\mathcal{B}_0}(Z_1),$$

$$\text{for} \quad Z_0 \in \partial\mathcal{B}_0, \quad t \in \mathbb{R}, \quad \text{and} \quad n \in \mathbb{N};$$

note that, by the summation convention, for $n \geq 2$,

$$\prod_{\ell=0}^{n-1} L_{Z_\ell}^{i_\ell i_{\ell+1}}(Z_{\ell+1}) = \sum_{i_1=1}^{3} \cdots \sum_{i_{n-1}=1}^{3} L_{Z_0}^{i_0 i_1}(Z_1) L_{Z_1}^{i_1 i_2}(Z_2) \cdots L_{Z_{n-1}}^{i_{n-1} i_n}(Z_n).$$

Finally, again for $\tilde{\mu} \in C_4^{\infty}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$, if j and n are positive integers, then $D_4^j\mathbb{L}^n\tilde{\mu}$ exists and is locally Hölder continuous on $\partial\mathcal{B}_0\times\mathbb{R}$, with

$$D_4^j\mathbb{L}^n\tilde{\mu} = \mathbb{L}^n D_4^j\tilde{\mu}. \tag{28}$$

(iv) Regarding L as defined on $\mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$, and $\mathbb{L}$ and Λ as defined on $\mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$, let us satisfy ourselves that

$$L\colon\ \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R}) \to \mathcal{E}^{H}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R}), \tag{29}$$

$$\mathbb{L}:\quad \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3) \to \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3), \tag{30}$$

and

$$\Lambda:\quad \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3) \to \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R}). \tag{31}$$

Suppose that $\mu \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$: by (i) and (ii), $L\mu \in C^\infty_4(\partial\mathcal{B}_0\times\mathbb{R})$ and is locally Hölder continuous (along with $D^j_4 L\mu$, for each $j \in \mathbb{N}$). Since μ vanishes on $\partial\mathcal{B}_0\times(-\infty,0]$, it is obvious from (11) and the form of $[\mu]_{(Z,\zeta)}$ for $Z \in \partial\mathcal{B}_0$ and $\zeta \in \mathbb{R}$ (cf., (II.1.2)) that $L\mu$ also vanishes on $\partial\mathcal{B}_0\times(-\infty,0]$. Consequently, to secure the inclusion $L\mu \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$, we must verify that the estimates required in (II.2.2) are fulfilled by $\{D^j_4 L\mu\}^\infty_{j=1}$. Choose $T > 0$ and $j \in \mathbb{N}$; if $Z \in \partial\mathcal{B}_0$ and $\zeta \in (0,T]$, then

$$\begin{aligned}
|(D^j_4 L\mu)(Z,\zeta)| &= |(LD^j_4\mu)(Z,\zeta)| \\
&= \frac{1}{2\pi}\left|\int_{\partial\mathcal{B}_0} L_Z\cdot\{[D^j_4\mu]_{(Z,\zeta)} + r^c_Z[D^{j+1}_4\mu]_{(Z,\zeta)}\}\, d\lambda_{\partial\mathcal{B}_0}\right| \\
&\le \frac{1}{2\pi}\left\{\int_{\partial\mathcal{B}_0} \frac{1}{r^2_Z}\cdot|r_{Z,k}\nu^k(Z)|\cdot|[D^j_4\mu]_{(Z,\zeta)}|\, d\lambda_{\partial\mathcal{B}_0}\right. \\
&\qquad \left. + \frac{1}{c}\int_{\partial\mathcal{B}_0} \frac{1}{r_Z}\cdot|r_{Z,k}\nu^k(Z)|\cdot|[D^{j+1}_4\mu]_{(Z,\zeta)}|\, d\lambda_{\partial\mathcal{B}_0}\right\} \\
&\le \frac{1}{2\pi}\, b_{\mu,T}\cdot C^j_{\mu,T}\cdot\left\{j^{(1+\delta_{\mu,T})j}\cdot\int_{\partial\mathcal{B}_0}\frac{1}{r^2_Z}|r_{Z,k}\nu^k(Z)|\, d\lambda_{\partial\mathcal{B}_0}\right. \\
&\qquad \left. + (j+1)^{(1+\delta_{\mu,T})(j+1)}\cdot\int_{\partial\mathcal{B}_0}\frac{1}{r_Z}\, d\lambda_{\partial\mathcal{B}_0}\right\}
\end{aligned}$$

$$< \frac{1}{2\pi} b_{\mu,T} \cdot 2^{(1+\delta_{\mu,T})} \cdot \left\{ \int_{\partial B_0} \frac{1}{r_Z^2} |r_{Z,k}\nu^k(Z)| \, d\lambda_{\partial B_0} + \int_{\partial B_0} \frac{1}{r_Z} \, d\lambda_{\partial B_0} \right\} \cdot \{(2e)^{(1+\delta_{\mu,T})} \cdot C_{\mu,T}\}^j \cdot j^{(1+\delta_{\mu,T})j}, \quad (32)$$

since, for $\alpha > 0$,

$$(j+1)^{\alpha(j+1)} \leq (2j)^{\alpha(j+1)} = 2^{\alpha} \cdot 2^{\alpha j} \cdot j^{\alpha} \cdot j^{\alpha j} < 2^{\alpha} \cdot (2e)^{\alpha j} \cdot j^{\alpha j},$$

having noted that $j^{\alpha} = e^{\alpha \ln j} < e^{\alpha j}$. It is easy to show that each of the integrals in (32) can be bounded independently of $Z \in \partial B_0$. For example, if $(a,1,d)$ is a set of Lyapunov constants for B_0^o,

$$\int_{\partial B_0} \frac{1}{r_Z^2} |r_{Z,k}\nu^k(Z)| \, d\lambda_{\partial B_0}$$

$$\leq \int_{\partial B_0 \cap B_d^3(Z)'} \frac{1}{r_Z^2} \, d\lambda + \int_{\partial B_0 \cap B_d^3(Z)} \frac{1}{r_Z^2} |r_{Z,k}\nu^k| \, d\lambda_{\partial B_0}$$

$$+ \int_{\partial B_0 \cap B_d^3(Z)} \frac{1}{r_Z^2} |\nu(Z)-\nu|_3 \, d\lambda_{\partial B_0}$$

$$\leq \frac{1}{d^2} \cdot \lambda_{\partial B_0}(\partial B_0) + (\hat{a}+a) \int_{\partial B_0 \cap B_d^3(Z)} \frac{1}{r_Z} \, d\lambda_{\partial B_0}$$

$$\leq \frac{1}{d^2} \cdot \lambda_{\partial B_0}(\partial B_0) + (\hat{a}+a) \cdot 2^{3/2} \pi d.$$

Thus, from (32) we see that the partial derivatives $\{D_4^j L_\mu\}_{j=1}^{\infty}$ satisfy

the required inequalities, which completes the proof of (29). Similarly, one can demonstrate that $\mathbb{L}\tilde{\mu} \in \mathcal{E}^{H}_{4,0}(\partial B_0 \times \mathbb{R};\mathbb{K}^3)$ and $\Lambda\tilde{\mu} \in \mathcal{E}^{H}_{4,0}(\partial B_0 \times \mathbb{R})$ whenever $\tilde{\mu} \in \mathcal{E}^{H}_{4,0}(\partial B_0 \times \mathbb{R};\mathbb{K}^3)$, *i.e.*, that (30) and (31) are correct.

[II.4] R E M A R K S. (a) Using the operators introduced in [II.3], the systems (II.1.1) can be written concisely in the form

$$\left.\begin{aligned} \Psi - \lambda L\Psi &= \lambda F_\lambda + \lambda\Lambda\psi \qquad (1)_1 \\ \psi - \lambda\mathbb{L}\psi &= \lambda f_\lambda \qquad (1)_2 \end{aligned}\right\} \text{ on } \partial B_0 \times \mathbb{R},$$

for $\lambda = 1$ or -1, wherein we have set

$$F_1 := 2\nu^k E^{\iota k^c} \Big| \; \partial B_0 \times \mathbb{R}, \tag{2}$$

$$F_{-1} := 2\nu^k B^{\iota k} \Big| \; \partial B_0 \times \mathbb{R}, \tag{3}$$

$$f^i_1 := 2\varepsilon_{ijk}\nu^j B^{\iota k} \Big| \; \partial B_0 \times \mathbb{R}, \tag{4}$$

and

$$f^i_{-1} := -2\varepsilon_{ijk}\nu^j E^{\iota k^c} \Big| \; \partial B_0 \times \mathbb{R}. \tag{5}$$

We are then led to examine separately the single equation

$$\Psi - \lambda L\Psi = F \quad \text{on} \quad \partial B_0 \times \mathbb{R} \tag{6}$$

and the system

$$\psi - \lambda\mathbb{L}\psi = f \quad \text{on} \quad \partial B_0 \times \mathbb{R}, \tag{7}$$

which we shall do in [II.7], subsequently applying the results to the system of ultimate interest, (1).

(b) The study of (6) and (7) is greatly expedited with the imposition of the following global geometric condition on $\mathcal{B}_0$ (as usual, we consider a null motion $\mathcal{M} \in \mathbb{M}(2)$); let (a,l,d) denote a set of Lyapunov constants for $\mathcal{B}_0^o$:

$$\text{(G)} \begin{cases} \text{there exists a positive number } a_0 \text{ such that} \\ \text{whenever } Z \in \partial\mathcal{B}_0 \text{ and } d \leq \rho_1 < \rho_2, \\ \displaystyle\int_{\partial\mathcal{B}_0 \cap B^3_{\rho_2}(Z) \cap B^3_{\rho_1}(Z)'} d\lambda_{\partial\mathcal{B}_0} \leq a_0(\rho_2-\rho_1). \end{cases} \tag{8}$$

This is essentially the hypothesis employed by Fulks and Guenther [17], who point out that it is fulfilled by a fairly large family of domains. Observe that if $\mathcal{B}_0$ satisfies (G), then no point of $\partial\mathcal{B}_0$ can be the center of a ball whose boundary contains a subset of $\partial\mathcal{B}_0$ of positive $\lambda_{\partial\mathcal{B}_0}$-measure (if $Z \in \partial\mathcal{B}_0$ were the center of such a ball of radius ρ_1, then obviously (8) would fail to hold for all $\rho_2 > \rho_1$ with $\rho_2-\rho_1$ sufficiently small).

Use of condition (G) leads to the following facts, which are simple variants of a result presented by Fulks and Guenther.

[II.5] L E M M A. *Let* $\mathcal{M}$ *be a null motion in* $\mathbb{M}(2)$ *for which* $\mathcal{B}_0$ *satisfies condition* (G). *There exists a positive number* α_0 *such that whenever* $\varepsilon > 0$, g *is a nonnegative continuous function*

on $[0,\varepsilon]$, *and* $Z \in \partial\mathcal{B}_0$,

$$\int_{\partial\mathcal{B}_0\cap B^3_\rho(Z)} \left\{\sum_{j=1}^{3} (L_Z^{ij})^2\right\}^{1/2} \cdot g\circ r_Z \, d\lambda_{\partial\mathcal{B}_0} \leq \alpha_0 \int_0^\rho g \, d\lambda_1, \tag{1}$$

$$\int_{\partial\mathcal{B}_0\cap B^3_\rho(Z)} \left\{\sum_{i=1}^{3}\sum_{j=1}^{3} (L_Z^{ij})^2\right\}^{1/2} \cdot g\circ r_Z \, d\lambda_{\partial\mathcal{B}_0} \leq \alpha_0 \int_0^\rho g \, d\lambda_1, \tag{2}$$

and

$$\int_{\partial\mathcal{B}_0\cap B^3_\rho(Z)} |L_Z|\cdot g\circ r_Z \leq \alpha_0 \int_0^\rho g \, d\lambda_1. \tag{3}$$

P R O O F. Let $(a,1,d_0)$ denote Lyapunov constants for $\mathcal{B}_0^o$, and $d \in (0,d_0)$. To prove (1), choose $Z \in \partial\mathcal{B}_0$. From (II.3.15), for each $Y \in \partial\mathcal{B}_0\setminus\{Z\}'$,

$$\begin{aligned}
\sum_{j=1}^{3} (L_Z^{ij}(Y))^2 &= \frac{1}{r_Z^4(Y)} \sum_{j=1}^{3} \{\{r_{Z,k}(Y)\nu^k(Z)\}^2\delta^{ij} \\
&\quad +2\{r_{Z,k}(Y)\nu^k(Z)\}\delta^{\underline{i}j}\cdot r_{Z,\underline{i}}(Y)\cdot\{\nu^j(Y)-\nu^j(Z)\} \\
&\quad +\{r_{Z,i}(Y)\cdot\{\nu^j(Y)-\nu^j(Z)\}\}^2\} \\
&= \frac{1}{r_Z^4(Y)} \{\{r_{Z,k}(Y)\nu^k(Z)\}^2+2\{r_{Z,k}(Y)\nu^k(Z)\}\cdot r_{Z,\underline{i}}(Y) \\
&\quad \cdot\{\nu^{\underline{i}}(Y)-\nu^{\underline{i}}(Z)\}+\{r_{Z,i}(Y)\}^2\cdot|\nu(Y)-\nu(Z)|_3^2\} \\
&\leq \frac{1}{r_Z^4(Y)} \{\{r_{Z,k}(Y)\nu^k(Z)\}^2+2|r_{Z,k}(Y)\nu^k(Z)|\cdot a r_Z(Y) \\
&\quad +a^2 r_Z^2(Y)\},
\end{aligned}$$

while

$$|r_{Z,k}(Y)\nu^k(Z)| \le |r_{Z,k}(Y)\nu^k(Y)|+|r_{Z,k}(Y)\cdot\{\nu^k(Z)-\nu^k(Y)\}|$$

$$\le |r_{Z,k}(Y)\nu^k(Y)|+a\cdot r_Z(Y),$$

so that

$$\left\{\sum_{j=1}^{3} (L_Z^{ij}(Y))^2\right\}^{1/2} \le \frac{1}{r_Z^2(Y)} \{\{a\cdot r_Z(Y)+|r_{Z,k}(Y)\nu^k(Y)|\}^2$$
$$+2a\cdot r_Z(Y)\cdot\{a\cdot r_Z(Y)+|r_{Z,k}(Y)\nu^k(Y)|\}+a^2r_Z^2(Y)\}^{1/2}$$
$$= \frac{1}{r_Z(Y)}\left\{2a+\frac{|r_{Z,k}(Y)\nu^k(Y)|}{r_Z(Y)}\right\}. \tag{4}$$

This gives

$$\left\{\sum_{j=1}^{3} (L_Z^{ij}(Y))^2\right\}^{1/2}$$

$$\le \begin{cases} \dfrac{1}{r_Z(Y)}(2a+\hat{a}) & , \quad \text{if} \quad Y\in\partial\mathcal{B}_0\cap B_d^3(Z)\cap\{Z\}', \\[2ex] \dfrac{1}{r_Z(Y)}\left\{2a+\dfrac{1}{d}\right\} \le \dfrac{1+2ad}{d^2} & , \quad \text{if} \quad Y\in\partial\mathcal{B}_0\cap B_d^3(Z)'. \end{cases} \tag{5}$$

Now, let $\rho > 0$, and suppose $g\colon\ [0,\rho] \to [0,\infty)$ is continuous. Consider first the case in which $\rho > d$: write

$$\int_{\partial\mathcal{B}_0\cap B_\rho^3(Z)} \left\{\sum_{j=1}^{3} (L_Z^{ij})^2\right\}^{1/2}\cdot g\circ r_Z \, d\lambda_{\partial\mathcal{B}_0} = I_1(Z,\rho)+I_2(Z,\rho), \tag{6}$$

wherein

$$I_1(Z) := \int_{\partial B_0 \cap B^3_d(Z)} \left\{ \sum_{j=1}^{3} (L_Z^{ij})^2 \right\}^{1/2} \cdot g \circ r_Z \; d\lambda_{\partial B_0}, \tag{7}$$

and

$$I_2(Z) := \int_{\partial B_0 \cap B^3_\rho(Z) \cap B^3_d(Z)'} \left\{ \sum_{i=1}^{3} (L_Z^{ij})^2 \right\}^{1/2} \cdot g \circ r_Z \; d\lambda_{\partial B_0}. \tag{8}$$

To estimate $I_1(Z)$, we begin by noting that, provided d_0 is sufficiently small (and $0 < d < d_0$), there exists a positive a_0' such that

$$\int_{\partial B_0 \cap B^3_{\zeta_2}(\hat{Z}) \cap B^3_{\zeta_1}(\hat{Z})'} \frac{1}{r_{\hat{Z}}} \, d\lambda_{\partial B_0} \leq a_0'(\zeta_2 - \zeta_1) \tag{9}$$

$$\text{whenever} \quad \hat{Z} \in \partial B_0 \quad \text{and} \quad 0 < \zeta_1 < \zeta_2 \leq d;$$

at the end of the proof, we shall verify that this is so. Select a partition, $\{\rho_k\}_{k=0}^N$, of $[0,d]$: $0 = \rho_0 < \rho_1 < \ldots < \rho_{N-1} < \rho_N = d$. Then, with

$$M_k(g) := \sup\{g(\zeta) \mid \rho_{k-1} \leq \zeta \leq \rho_k\}, \quad \text{for each} \quad k \in \{1,\ldots,N\}, \tag{10}$$

we have, by (5) and (9),

$$I_1(Z) = \sum_{k=1}^{N} \int_{\partial B_0 \cap B^3_{\rho_k}(Z) \cap B^3_{\rho_{k-1}}(Z)'} \left\{ \sum_{j=1}^{3} (L_Z^{ij})^2 \right\}^{1/2} \cdot g \circ r_Z \; d\lambda_{\partial B_0}$$

$$\leq (2a+\hat{a})\cdot \sum_{k=1}^{N} M_k(g)\cdot \int_{\partial B_0 \cap B^3_{\rho_k}(Z) \cap B^3_{\rho_{k-1}}(Z)'} \frac{1}{r_Z}\, d\lambda_{\partial B_0}$$

$$\leq (2a+\hat{a})\cdot a_0'\cdot \sum_{k=0}^{N} M_k(g)\cdot(\rho_k-\rho_{k-1}) \tag{11}$$

$$= (2a+\hat{a})\cdot a_0'\cdot \int_0^d g\, d\lambda_1$$

$$+(2a+\hat{a})\cdot a_0'\left\{\sum_{k=0}^{N} M_k(g)\cdot(\rho_k-\rho_{k-1}) - \int_0^d g\, d\lambda_1\right\}.$$

Thus,

$$I_1(Z) \leq (2a+\hat{a})\cdot a_0'\cdot \int_0^d g\, d\lambda_1, \tag{12}$$

since the second term on the right in (11) can be made arbitrarily small by choosing a partition of sufficiently small norm. In examining $I_2(Z)$, we invoke condition (G) (cf., (II.4.8)): for any partition, $\{\rho_k\}_{k=0}^N$, of $[d,\rho]$, with $d = \rho_0 < \rho_1 < \ldots < \rho_{N-1} < \rho_N = \rho$, defining $\{M_k(g)\}_{k=1}^N$ as in (10), and again using (5), we find that

$$I_2(Z) = \sum_{k=1}^{N} \int_{\partial B_0 \cap B^3_{\rho_k}(Z) \cap B^3_{\rho_{k-1}}(Z)'} \left\{\sum_{j=1}^{3} (L_Z^{ij})^2\right\}^{1/2}\cdot g\circ r_Z\, d\lambda_{\partial B_0}$$

$$\leq \frac{1+2ad}{d^2} \sum_{k=1}^{N} M_k(g)\cdot \int_{\partial B_0 \cap B^3_{\rho_k}(Z) \cap B^3_{\rho_{k-1}}(Z)'} d\lambda_{\partial B_0}$$

$$\leq \frac{1+2ad}{d^2}\cdot a_0\cdot \sum_{k=1}^{N} M_k(g)\cdot(\rho_k-\rho_{k-1})$$

$$= \frac{1+2ad}{d^2} \cdot a_0 \cdot \int_d^\rho g \, d\lambda_1$$

$$+ \frac{1+2ad}{d^2} \cdot a_0 \cdot \left\{ \sum_{k=1}^{N} M_k(g) \cdot (\rho_k - \rho_{k-1}) - \int_d^\rho g \, d\lambda_1 \right\} .$$

Reasoning as before, this implies that

$$I_2(Z) \leq \frac{1+2ad}{d^2} \cdot a_0 \cdot \int_d^\rho g \, d\lambda_1 . \tag{13}$$

From (6), (12), and (13),

$$\int_{\partial\mathcal{B}_0 \cap B^3_\rho(Z)} \left\{ \sum_{j=1}^{3} (L_Z^{ij})^2 \right\}^{1/2} \cdot g \circ r_Z \, d\lambda_{\partial\mathcal{B}_0}$$
$$\leq \max \left\{ a_0' \cdot (2a+\hat{a}), \;\; a_0 \cdot \frac{1+2ad}{d^2} \right\} \cdot \int_0^\rho g \, d\lambda_1 . \tag{14}$$

If $0 < \rho \leq d$, then (14) still holds, for, then we need only effect an estimate of the type already carried out for $I_1(Z)$; for this, note that we have no need to appeal to hypothesis (G).

Again with $Z \in \partial\mathcal{B}_0$, a computation of the same sort leading to (4) and (5) produces the inequalities

$$\left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} (L_Z^{ij}(Y))^2 \right\}^{1/2}$$

$$\leq \begin{cases} \dfrac{1}{r_Z(Y)} \{6a^2+8a\hat{a}+3\hat{a}^2\}^{1/2} , & \text{if } Y \in \partial\mathcal{B}_0 \cap B^3_d(Z) \setminus \{Z\} , \\[2ex] \dfrac{1}{r_Z(Y)} \left\{ 6a^2 + \dfrac{8a}{d} + \dfrac{3}{d^2} \right\}^{1/2} \leq \dfrac{1}{d} \left\{ 6a^2 + \dfrac{8a}{d} + \dfrac{3}{d^2} \right\}^{1/2} , & \text{if } Y \in \partial\mathcal{B}_0 \cap B^3_d(Z)' , \end{cases} \tag{15}$$

while it is easy to see that

$$|L_Z(Y)| = \frac{1}{r_Z^2(Y)} \cdot |r_{Z,k}(Y)\nu^k(Z)|$$

$$\leq \begin{cases} \frac{1}{r_Y(Z)} \cdot (a+\hat{a}) & , \quad \text{if} \quad Y \in \partial B_0 \cap B_d^3(Z) \sim \{Z\}', \\ \frac{1}{r_Y(Z)} \cdot \left(a + \frac{1}{d}\right) \leq \frac{1+ad}{d^2} & , \quad \text{if} \quad Y \in \partial B_0 \cap B_d^3(Z)'. \end{cases} \tag{16}$$

Using (15) and (16), we can construct an argument like that which produced (14) in order to conclude that there exist $a_1 > 0$ and $a_2 > 0$, depending only upon B_0, for which

$$\int_{\partial B_0 \cap B_\rho^3(Z)} \left\{\sum_{i=1}^{3} \sum_{j=1}^{3} (L_Z^{ij})^2\right\}^{1/2} \cdot g \circ r_Z \, d\lambda_{\partial B_0} \leq a_1 \cdot \int_0^\rho g \, d\lambda_{\partial B_0}, \tag{17}$$

and

$$\int_{\partial B_0 \cap B_\rho^3(Z)} |L_Z| \cdot g \circ r_Z \, d\lambda_{\partial B_0} \leq a_2 \cdot \int_0^\rho g \, d\lambda_{\partial B_0}, \tag{18}$$

$$\text{for} \quad Z \in \partial B_0 \quad \text{and} \quad \rho > 0,$$

whenever $g: [0,\rho] \to [0,\infty)$ is continuous.

In view of (14), (17), and (18), the existence of a positive number α_0 possessing the required properties shall follow once we have shown that there is an $a_0' > 0$ such that (9) is true, provided $0 < d < d_0$ and d_0 is sufficiently small. Select $Z \in \partial B_0$ and set

$$I_Z(\rho) := \int_{\partial\mathcal{B}_0 \cap B^3_\rho(Z)} \frac{1}{r_Z}\, d\lambda_{\partial\mathcal{B}_0}$$

$$= \int_{h_Z(\partial\mathcal{B}_0 \cap B^3_\rho(Z))} \frac{1}{r_Z \circ h_Z^{-1}} \cdot Jh_Z^{-1}\, d\lambda_2 \quad \text{for} \quad 0 < \rho < d_0; \tag{19}$$

since

$$I_Z(\rho_2) - I_Z(\rho_1) = \int_{\partial\mathcal{B}_0 \cap B^3_{\rho_2}(Z) \cap B^3_{\rho_1}(Z)'} \frac{1}{r_Z}\, d\lambda_{\partial\mathcal{B}_0} \quad \text{for} \quad 0 < \rho_1 < \rho_2 < d_0,$$

in order to prove (9) it suffices to show that I_Z possesses a derivative on $(0,d_0)$ which is bounded uniformly in Z. Now, for each $\rho \in (0,d_0)$, we know that $h_Z(\partial\mathcal{B}_0 \cap B^3_\rho(Z))$ is starlike with respect to $0 \in \mathbb{R}^2$ and coincides as a subset of $\mathbb{R}^3$ with the projection onto $\{Y \in \mathbb{R}^3 |\ Y^3 = 0\}$ of the intersection of $B^3_\rho(0)$ and the graph of a function $f_Z \in C^2(h_Z(\partial\mathcal{B}_0 \cap B^3_{d_0}(Z)))$; the boundary $\partial\{h_Z(\partial\mathcal{B}_0 \cap B^3_\rho(Z))\}$ is also starlike with respect to 0 and can be identified as the projection of the intersection of $\partial B^3_\rho(0)$ with the graph of f_Z. Thus, there is a 2π-periodic function $R(\rho,\cdot)$ on $\mathbb{R}$ such that $\partial\{h_Z(\partial\mathcal{B}_0 \cap B^3_\rho(Z))\}$ is described in polar coordinates with pole at the origin as the set $\{(R(\rho,\theta),\theta) |\ 0 \le \theta < 2\pi\}$. Clearly, if we let σ denote the map $(s,\theta) \mapsto (s\cos\theta,\ s\sin\theta)$ on $(0,\infty)\times(0,2\pi)$ and define $\hat{f}$ on the open set $\sigma^{-1}(h_Z(\partial\mathcal{B}_0 \cap B^3_{d_0}(Z)))$ *via*

$$\hat{f} := f \circ \sigma,$$

we have

$$R^2(\rho,\theta)+\hat{f}^2(R(\rho,\theta),\theta) = \rho^2 \quad \text{for} \quad 0 < \rho < d_0 \quad \text{and} \quad 0 < \theta < 2\pi. \tag{20}$$

Since $\hat{f}$ is of class C^2, it follows from (20) and the implicit function theorem that $R \in C^2((0,d_0)\times(0,2\pi))$. Then (20) also gives

$$\{R(\rho,\theta)+\hat{f}(R(\rho,\theta),\theta)\cdot\hat{f}_{,1}(R(\rho,\theta),\theta)\}\cdot R_{,1}(\rho,\theta) = \rho,$$

whence we must also have

$$R_{,1}(\rho,\theta) = \frac{\rho}{R(\rho,\theta)+\hat{f}(R(\rho,\theta),\theta)\cdot\hat{f}_{,1}(R(\rho,\theta),\theta)} \tag{21}$$

$$\text{for} \quad 0 < \rho < d_0 \quad \text{and} \quad 0 < \theta < 2\pi.$$

Using [VI.64.iii.2 and 4], it is easy to see that

$$\begin{aligned}|\hat{f}(R(\rho,\theta),\theta)\cdot\hat{f}_{,1}(R(\rho,\theta),\theta)| &\leq \tilde{a}\cdot R^2(\rho,\theta)\cdot \frac{8}{7} a\rho \\ &< \frac{8}{7} a\tilde{a}\cdot d_0^2\cdot R(\rho,\theta)\end{aligned} \tag{22}$$

$$\text{for} \quad 0 < \rho < d_0 \quad \text{and} \quad 0 < \theta < 2\pi,$$

since

$$\begin{aligned}|\hat{f}_{,1}(s,\theta)| &= |\cos\theta\cdot f_{,1}(s\cdot\cos\theta,\ s\cdot\sin\theta) \\ &\quad +\sin\theta\cdot f_{,2}(s\cdot\cos\theta,\ s\cdot\sin\theta)| \\ &\leq |\operatorname{grad} f\,(s\cdot\cos\theta,\ s\cdot\sin\theta)|_2.\end{aligned}$$

Supposing now that d_0 is so small that, say,

$$\frac{8}{7}\, a\tilde{a}\cdot d_0^2 \leq \frac{1}{2}\,, \tag{23}$$

and observing from [VI.64.iii.6] that

$$R(\rho,\theta) \geq \frac{7}{9}\,\rho, \qquad \text{for} \qquad 0 < \rho < d_0 \qquad \text{and} \qquad 0 < \theta < 2\pi,$$

(21)-(23) give

$$0 < R,_1(\rho,\theta) < \frac{\rho}{R(\rho,\theta) - \frac{1}{2}\,R(\rho,\theta)} \leq \frac{18}{7} \tag{24}$$

$$\text{for} \qquad 0 < \rho < d_0 \qquad \text{and} \qquad 0 < \theta < 2\pi.$$

The starlike nature of each set $h_Z(\partial\mathcal{B}_0 \cap B_\rho^3(Z))$, for $\rho \in (0,d_0)$, and the properties of R show that

$$I_Z(\rho) = \int_0^{2\pi} \int_0^{R(\rho,\theta)} \left\{\frac{1}{r_Z \circ h_Z^{-1}} \cdot \mathcal{J}h_Z^{-1}\right\} \circ \sigma(s,\theta)\cdot s\; ds\; d\theta, \tag{25}$$

$$\text{for each} \qquad \rho \in (0,d_0).$$

We have

$$r_Z \circ h_Z^{-1}(\hat{\xi}) = r_Z(\pi_Z^{-1}(\hat{\mathcal{H}}_Z^{-1}(\hat{\xi}))) \geq r_Z(\hat{\mathcal{H}}_Z^{-1}(\hat{\xi})) = |\hat{\xi}|_2$$

$$\text{if} \qquad \hat{\xi} \in h_Z(\partial\mathcal{B}_0 \cap B_{d_0}^3(Z)),$$

so

$$r_Z \circ h_Z^{-1} \circ \sigma(s,\theta) \geq s \qquad \text{if} \qquad \theta \in (0,2\pi) \qquad \text{and} \qquad s \in (0,R(\rho,\theta)),$$

$$\text{wherein} \qquad \rho \in (0,d_0).$$

Therefore, recalling that $Jh_Z^{-1} \leq \sqrt{2}$ on $h_Z(\partial\mathcal{B}_0 \cap B^3_{d_0}(Z))$, we can conclude that the integrand in (25) is majorized by $\sqrt{2}$ whenever $Z \in \partial\mathcal{B}_0$ and $\rho \in (0,d_0)$. We may then assert that I_Z' exists on $(0,d_0)$ and compute, using (24) and (25),

$$I_Z'(\rho) = \int_0^{2\pi} R,_1(\rho,\theta)\cdot\left\{\frac{1}{r_Z \circ h_Z^{-1}}\cdot Jh_Z^{-1}\right\}\circ\sigma(R(\rho,\theta),\theta)\cdot R(\rho,\theta)\ d\theta$$

$$\leq \frac{18}{7}\cdot\sqrt{2}\cdot\int_0^{2\pi} d\theta$$

$$= \frac{18}{7}\cdot\sqrt{2}\cdot 2\pi, \qquad \text{for} \quad \rho \in (0,d_0) \quad \text{and} \quad Z \in \partial\mathcal{B}_0.$$

As we have remarked, the existence of I_Z' on $(0,d_0)$ for each $Z \in \partial\mathcal{B}_0$ and its uniform boundedness in Z show that the Lipschitz condition (9) holds, provided that d_0 is chosen as in (23); in fact, we can take

$$a_0' = \frac{36\pi}{7}\cdot\sqrt{2}\ .$$

This completes the proof of the lemma. □.

[II.6] R E M A R K. If, in [II.5], it is not required that $\mathcal{B}_0$ satisfy condition (G), then the conclusions of that lemma still hold for $\rho \in [0,d]$, wherein d is chosen as in the proof of the lemma. This follows from an inspection of the proof presented.

As promised, we proceed to state and prove existence results for (II.4.6) and (II.4.7).

[II.7] T H E O R E M. *Let* M *be a null motion in* $\mathbb{M}(2)$ *for which* $\mathcal{B}_0$ *satisfies condition* (G). *Let* $\lambda \in \mathbb{K}$, $\lambda \neq 0$.

(*i*) *Suppose that* $f \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$.

(*i.a*) *There exists a function* $\psi \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$ *such that*

$$\psi-\lambda\mathbb{L}\psi = f \quad \text{on} \quad \partial\mathcal{B}_0\times\mathbb{R}. \tag{1}$$

In fact, the function

$$\psi := \sum_{n=0}^{\infty} \lambda^n\mathbb{L}^n f \tag{2}$$

has this property, the series converging absolutely on $\partial\mathcal{B}_0\times\mathbb{R}$ *and uniformly on each compact subset of* $\partial\mathcal{B}_0\times\mathbb{R}$.

(*i.b*) *With* ψ *given by* (2),

$$D_4^j\psi = \sum_{n=0}^{\infty} \lambda^n\mathbb{L}^n D_4^j f \quad \text{on} \quad \partial\mathcal{B}_0\times\mathbb{R} \qquad \text{for each} \quad j \in \mathbb{N}, \tag{3}$$

each series converging absolutely on $\partial\mathcal{B}_0\times\mathbb{R}$ *and uniformly on each compact subset of* $\partial\mathcal{B}_0\times\mathbb{R}$. *For each* $j \in \mathbb{N}\cup\{0\}$ *and* $T > 0$,[+]

[+]We employ the convention $0^0 := 1$.

$$|D_4^j\psi^i| \le \tilde{b}_{\psi,T}\cdot C^j_{\psi,T}\cdot j^{(1+\delta_{\psi,T})j} \quad on \quad \partial B_0\times[0,T], \tag{4}$$

wherein

$$\tilde{b}_{\psi,T} := b_{f,T} \sum_{n=0}^{\infty} \left\{\frac{1}{2\pi}(1+2^{1+\delta_{f,T}}\cdot C_{f,T}T)|\lambda|\cdot\alpha_0 cT\right\}^n \cdot \frac{n^{\delta_{f,T}n}}{n!} \tag{5}$$

(with α_0 *as in* [II.5]),

$$C_{\psi,T} := 2^{1+\delta_{f,T}}\cdot C_{f,T}, \tag{6}$$

and

$$\delta_{\psi,T} := \delta_{f,T}. \tag{7}$$

In particular, for each $j \in \mathbb{N}\cup\{0\}$,

$$|D_4^j\psi^i(Z,\zeta)| \le \tilde{b}_{\psi,\zeta}C^j_{\psi,\zeta}\cdot j^{(1+\delta_{\psi,\zeta})j} \quad for \quad Z\in\partial B_0, \quad \zeta > 0. \tag{8}$$

(*i.c*) *If* $f \in \mathcal{E}^H_{4,0}(\partial B_0\times\mathbb{R};\mathbb{K}^3)$, *then* $\psi \in \mathcal{E}^H_{4,0}(\partial B_0\times\mathbb{R};\mathbb{K}^3)$.

(*ii*) *Suppose that* $F \in \mathcal{E}_{4,0}(\partial B_0\times\mathbb{R})$.

(*ii.a*) *There exists a function* $\Psi \in \mathcal{E}_{4,0}(\partial B_0\times\mathbb{R})$ *such that*

$$\Psi-\lambda L\Psi = F \quad on \quad \partial B_0\times\mathbb{R}. \tag{9}$$

In fact, the function

$$\Psi := \sum_{n=0}^{\infty} \lambda^n L^n F$$

has this property, the series converging absolutely on $\partial B_0 \times \mathbb{R}$ *and uniformly on each compact subset of* $\partial B_0 \times \mathbb{R}$.

(*ii.b*) *With* Ψ *given by* (9),

$$D_4^j \Psi = \sum_{n=0}^{\infty} \lambda^n L^n D_4^j F \quad \text{on} \quad \partial B_0 \times \mathbb{R} \quad \text{for each} \quad j \in \mathbb{N}, \tag{10}$$

each series converging absolutely on $\partial B_0 \times \mathbb{R}$ *and uniformly on each compact subset of* $\partial B_0 \times \mathbb{R}$. *For each* $j \in \mathbb{N} \cup \{0\}$ *and* $T > 0$,

$$|D_4^j \Psi| \le b_{\Psi,T} \cdot C_{\Psi,T}^j \cdot j^{(1+\delta_{\Psi,T})j} \quad \text{on} \quad \partial B_0 \times [0,T], \tag{11}$$

wherein

$$b_{\Psi,T} := b_{F,T} \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi} \left(1+2^{1+\delta_{F,T}} \cdot C_{F,T} T\right) |\lambda| \alpha_0 c T \right\}^n \cdot \frac{n^{\delta_{F,T} n}}{n!} \tag{12}$$

(*with* α_0 *as in* [II.5]),

$$C_{\Psi,T} := 2^{1+\delta_{F,T}} \cdot C_{F,T}, \tag{13}$$

and

$$\delta_{\Psi,T} := \delta_{F,T}. \tag{14}$$

In particular, for each $j \in \mathbb{N}\cup\{0\}$,

$$|D_4^j\Psi(Z,\zeta)| \leq b_{\Psi,\zeta} C_{\Psi,\zeta}^j \cdot j^{(1+\delta_{\Psi,\zeta})j}$$

$$\text{for} \quad Z \in \partial\mathcal{B}_0, \quad \zeta > 0. \tag{15}$$

(ii.c) If $F \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$, *then* $\Psi \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$.

P R O O F. (i) We define the sequence $(\psi_n)_{n=0}^{\infty}$ (of "successive approximations") on $\partial\mathcal{B}_0\times\mathbb{R}$ according to

$$\psi_0 := f,$$

$$\psi_n := f+\lambda\mathbb{L}\psi_{n-1} \quad \text{for each} \quad n \in \mathbb{N};$$

by (II.3.30), each ψ_n lies in $\mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$. We have

$$\psi_1 = f+\lambda\mathbb{L}f,$$

$$\psi_2 = f+\lambda\mathbb{L}(f+\lambda\mathbb{L}f) = f+\lambda\mathbb{L}f+\lambda^2\mathbb{L}^2 f,$$

$$\ldots,$$

and an easy induction gives

$$\psi_n = \sum_{k=0}^{n} \lambda^k\mathbb{L}^k f \quad \text{for each} \quad n \in \mathbb{N}\cup\{0\},$$

just the partial sums of the formal series $\sum_{k=0}^{\infty} \lambda^k\mathbb{L}^k f$. Let us then examine the convergence properties of the latter, beginning by deriving an estimate for the modulus of the general term of the series. For

$k \in \mathbb{N}$ and points $Z_0,\ldots,Z_k$ chosen from ∂B_0, we define, if $t > 0$,

$$\partial B_0(Z_0;t) := \{Y \in \partial B_0 \mid \ t - r^c_{Z_0}(Y) > 0\}, \tag{16}$$

$$\partial B_0(Z_0,Z_1;t) := \{Y \in \partial B_0 \mid \ t - r^c_{Z_0}(Z_1) - r^c_{Z_1}(Y) > 0\},$$

$$\ldots,$$

and

$$\partial B_0(Z_0,\ldots,Z_k;t) := \{Y \in \partial B_0 \mid \ t - \sum_{j=0}^{k-1} r^c_{Z_j}(Z_{j+1}) - r^c_{Z_k}(Y) > 0\}. \tag{17}$$

Then, because f vanishes on $\partial B_0 \times (-\infty,0]$, from (II.3.27) it is clearly permissible to write

$$\{\mathbb{L}^n f\}^{i_0}(Z_0,t)$$

$$= \left(\frac{-1}{2\pi}\right)^n \int_{\partial B_0(Z_0;t)} \cdots \int_{\partial B_0(Z_0,\ldots,Z_{n-1};t)} \left\{\prod_{\ell=0}^{n-1} L_{Z_\ell}^{i_\ell i_{\ell+1}}(Z_{\ell+1})\right\}$$

$$\cdot\left\{\left(\prod_{j=0}^{n-1} \{1 + r^c_{Z_j}(Z_{j+1})D_4\}\right) f^{i_n}\right\}\left(Z_n, t - \sum_{k=0}^{n-1} r^c_{Z_k}(Z_{k+1})\right)$$

$$d\lambda_{\partial B_0}(Z_n) \ \cdots \ d\lambda_{\partial B_0}(Z_1)$$

$$\text{for} \quad Z_0 \in \partial B_0, \quad t > 0, \quad \text{and} \quad n \in \mathbb{N},$$

with which Cauchy's inequality produces

$$
\begin{aligned}
&\left|\{\mathbb{L}^n f\}^{i_0}(Z_0,t)\right| \\
&\leq \left(\frac{1}{2\pi}\right)^n \int_{\partial B_0(Z_0;T)} \cdots \int_{\partial B_0(Z_0,\ldots,Z_{n-1};t)} \\
&\cdot\left\{\sum_{i_n=1}^{3}\left\{\prod_{\ell=0}^{n-1} L_{Z_\ell}^{i_\ell i_{\ell+1}}(Z_{\ell+1})\right\}^2\right\}^{1/2} \\
&\cdot\left|\left\{\left(\prod_{j=0}^{n-1}\{1+r^c_{Z_j}(Z_{j+1})D_4\}\right)f\right\}\left(Z_n,t-\sum_{k=0}^{n-1} r^c_{Z_k}(Z_{k+1})\right)\right|_3 \\
&\qquad\qquad d\lambda_{\partial B_0}(Z_n)\ \cdots\ d\lambda_{\partial B_0}(Z_1)
\end{aligned}
\tag{18}
$$

for $Z_0 \in \partial B_0$, $t > 0$, and $n \in \mathbb{N}$.

Now, let $k \in \mathbb{N}$, and suppose that $\{\beta_j\}_{j=0}^k \subset \mathbb{K}$. We claim that the expansion

$$\prod_{j=0}^{k} (1+\beta_j D_4) = \sum_{j=0}^{k+1} \alpha_j^k(\beta_\ell) D_4^j \tag{19}$$

holds, with $\alpha_0^k(\beta_\ell) = 1$, and, for $j \in \{1,\ldots,k+1\}$, $\alpha_j^k(\beta_\ell)$ is a sum of $\binom{k+1}{j}$ terms, each term being a product of j of the β's. This is obviously the case for $k = 1$. Suppose $n \in \mathbb{N}$ and the c... is true for $k = n$. Then

$$
\begin{aligned}
\prod_{j=0}^{n+1} (1+\beta_j D_4) &= (1+\beta_{n+1}D_4)\cdot \prod_{j=0}^{n} (1+\beta_j D_4) \\
&= (1+\beta_{n+1}D_4)\cdot \sum_{j=0}^{n+1} \alpha_j^n(\beta_\ell) D_4^j
\end{aligned}
$$

$$= \alpha_0^n(\beta_\ell) + \sum_{j=1}^{n+1} \{\alpha_j^n(\beta_\ell) + \beta_{n+1}\cdot\alpha_{j-1}^n(\beta_\ell)\} D_4^j \qquad (20)$$

$$+\beta_{n+1}\cdot\alpha_{n+1}^n(\beta_\ell)\cdot D_4^{n+2},$$

so

$$\alpha_0^{n+1}(\beta_\ell) = \alpha_0^n(\beta_\ell) = 1,$$

$$\alpha_j^{n+1}(\beta_\ell) = \alpha_j^n(\beta_\ell) + \beta_{n+1}\cdot\alpha_{j-1}^n(\beta_\ell) \quad \text{for} \quad j \in \{1,\ldots,n+1\},$$

and

$$\alpha_{n+2}^{n+1}(\beta_\ell) = \beta_{n+1}^n\cdot\alpha_{n+1}^n(\beta_\ell).$$

Obviously, $\alpha_0^{n+1}(\beta_\ell)$ and $\alpha_{n+2}^{n+1}(\beta_\ell)$ are of the required form (note that $\alpha_{n+1}^n(\beta_\ell)$ has $\binom{n+1}{n+1} = 1$ term), while for $j \in \{1,\ldots,n+1\}$, $\alpha_j^{n+1}(\beta_\ell)$ has

$$\binom{n+1}{j} + \binom{n+1}{j-1} = \binom{n+2}{j}$$

terms, each term comprising a product of j of the β's. Thus, our claim is substantiated, (20) being of the required form for $k = n+1$.

Returning to the integrand of (18), let $n \in \mathbb{N}$ with $n \geq 2$, $t > 0$, and $Z_0,\ldots,Z_n \in \partial B_0$, with $t - \sum_{k=0}^{n-1} r_{Z_k}^c(Z_{k+1}) > 0$. Using the familiar inequality

$$\{a_1\ldots a_m\}^{1/m} \leq \frac{1}{m} \sum_{k=1}^{m} a_k$$

relating the geometric and arithmetic means of m nonnegative numbers $a_1,\dots,a_m$, if $\{i_k\}_{k=1}^{j}$ consists of distinct integers chosen from $\{0,\dots,n-1\}$ $(1 \le j \le n)$, we must have

$$\prod_{k=1}^{j} r^c_{Z_{i_k}}(Z_{i_k+1}) \le \left\{\frac{1}{j}\sum_{k=1}^{j} r^c_{Z_{i_k}}(Z_{i_k+1})\right\}^j < \left(\frac{t}{j}\right)^j. \tag{21}$$

Consequently, if $T > 0$ and $t \in (0,T]$, recalling the estimates for the 4-derivatives of $f \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$,

$$\begin{aligned}
&\left|\left\{\left(\prod_{j=0}^{n-1}\{1+r^c_{Z_j}(Z_{j+1})D_4\}\right)f\right\}\left(Z_n, t-\sum_{k=0}^{n-1} r^c_{Z_k}(Z_{k+1})\right)\right|_3 \\
&\le \sum_{j=0}^{n} \alpha_j^{n-1}(r^c_{Z_\ell}(Z_{\ell+1}))\cdot\left|D_4^j f\left(Z_n, t-\sum_{k=0}^{n-1} r^c_{Z_k}(Z_{k+1})\right)\right|_3 \\
&\le b_{f,T} + \sum_{j=1}^{n} \alpha_j^{n-1}(r^c_{Z_\ell}(Z_{\ell+1}))\cdot b_{f,T}C^j_{f,T}\cdot j^{(1+\delta_{f,T})j} \\
&\le b_{f,T} + \sum_{j=1}^{n} \binom{n}{j}\cdot\left(\frac{t}{j}\right)^j\cdot b_{f,T}C^j_{f,T}\cdot j^{(1+\delta_{f,T})j} \\
&\le b_{f,T}\cdot n^{\delta_{f,T}n}\cdot \sum_{j=0}^{n}\binom{n}{j}\cdot(C_{f,T}T)^j \\
&= b_{f,T}(1+C_{f,T}T)^n\cdot n^{\delta_{f,T}n}.
\end{aligned} \tag{22}$$

If $n = 1$, it is easy to check that the final estimate in (22) remains valid. Using this with (18), when $T > 0$ and $n \in \mathbb{N}$, we find

$$|\{\mathbb{L}^n f\}^{i_0}(Z_0,t)| \le b_{f,T}\cdot\left\{\frac{1}{2\pi}(1+C_{f,T}T)\right\}^n\cdot n^{\delta_{f,T}n}\cdot I_n^{i_0}(Z_0,t)$$
$$\text{for} \quad Z_0 \in \partial\mathcal{B}_0 \quad \text{and} \quad t \in (0,T], \tag{23}$$

having written

$$I_n^{i_0}(Z_0,t)$$

$$:= \int_{\partial B_0(Z_0;t)} \cdots \int_{\partial B_0(Z_0,\ldots,Z_{n-1};t)} \left\{\sum_{i_n=1}^{3} \left\{\prod_{\ell=0}^{n-1} L_{Z_\ell}^{i_\ell i_{\ell+1}}(Z_{\ell+1})\right\}^2\right\}^{1/2} \tag{24}$$

$$d\lambda_{\partial B_0}(Z_n) \ldots d\lambda_{\partial B_0}(Z_1).$$

We must next estimate the integrals given by (24); for this, we shall use the hypothesis that B_0 satisfies condition (G), and Lemma [II.5]. Let $T > 0$, and choose $Z_0 \in \partial B_0$ and $t \in (0,T]$. Directly from (II.5.1),

$$\begin{aligned} I_1^{i_0}(Z_0,t) &= \int_{\partial B_0 \cap \mathbb{B}^3_{ct}(Z_0)} \left\{\sum_{i_1=1}^{3} (L_{Z_0}^{i_0 i_1})^2\right\}^{1/2} d\lambda_{\partial B_0} \\ &\leq \alpha_0 \int_0^{ct} d\lambda_1 \\ &= \alpha_0 \cdot ct \\ &\leq \alpha_0 \cdot cT. \end{aligned} \tag{25}$$

If $n \geq 2$ and $\{a_k^{ij} \mid i,j = 1,2,3;\ k = 0,\ldots,n-1\} \subset \mathbb{R}$, then

$$\left\{\sum_{i_n=1}^{3} (a_0^{i_0 i_1} a_1^{i_1 i_2} \cdots a_{n-1}^{i_{n-1} i_n})^2\right\}^{1/2}$$

$$\leq \left\{\sum_{j_0=1}^{3} (a_0^{i_0 j_0})^2\right\}^{1/2} \cdot \left\{\sum_{i_1=1}^{3} \sum_{j_1=1}^{3} (a_1^{i_1 j_1})^2\right\}^{1/2} \cdots$$

$$\cdot \left\{\sum_{i_{n-1}=1}^{3} \sum_{j_{n-1}=1}^{3} (a_{n-1}^{i_{n-1} j_{n-1}})^2\right\}^{1/2};$$

this can be proven by induction, using Cauchy's inequality. Thus, for $n \geq 2$,

$$I_n^{i_0}(Z_0,t)$$

$$= \int_{\partial\mathcal{B}_0(Z_0;t)} \cdots \int_{\partial\mathcal{B}_0(Z_0,\ldots,Z_{n-1};t)} \left\{\sum_{i_n=1}^{3} \{L_{Z_0}^{i_0 i_1}(Z_1)\cdot L_{Z_1}^{i_1 i_2}(Z_2)\cdots L_{Z_{n-1}}^{i_{n-1} i_n}(Z_n)\}^2\right\}^{1/2}$$

$$d\lambda_{\partial\mathcal{B}_0}(Z_n) \cdots d\lambda_{\partial\mathcal{B}_0}(Z_1) \tag{26}$$

$$\leq \int_{\partial\mathcal{B}_0(Z_0;t)} \cdots \int_{\partial\mathcal{B}_0(Z_0,\ldots,Z_{n-1};t)} \left\{\sum_{j_0=1}^{3} \{L_{Z_0}^{i_0 j_0}(Z_1)\}^2\right\}^{1/2}$$

$$\cdot \left\{\sum_{i_1=1}^{3} \sum_{j_1=1}^{3} \{L_{Z_1}^{i_1 j_1}(Z_2)\}^2\right\}^{1/2} \cdots$$

$$\cdot \left\{\sum_{i_{n-1}=1}^{3} \sum_{j_{n-1}=1}^{3} \{L_{Z_{n-1}}^{i_{n-1} j_{n-1}}(Z_n)\}^2\right\}^{1/2} d\lambda_{\partial\mathcal{B}_0}(Z_n) \cdots d\lambda_{\partial\mathcal{B}_0}(Z_1).$$

Upon appealing to (II.5.2), we see that the innermost integral in (26),

taken over $\partial\mathcal{B}_0(Z_0,\ldots,Z_{n-1};t) = \partial\mathcal{B}_0 \cap B^3_{ct-\sum_{j=0}^{n-2} r_{Z_j}(Z_{j+1})}(Z_{n-1})$

(cf., (17)), is majorized by

$$\alpha_0 \cdot \int_0^{ct-\sum_{j=0}^{n-2} r_{Z_j}(Z_{j+1})} d\lambda_1 = \alpha_0 \cdot \left\{ ct - \sum_{j=0}^{n-2} r_{Z_j}(Z_{j+1}) \right\},$$

so that, for $n = 2$, using (II.5.1) again,

$$\begin{aligned} I_2^{i_0}(Z_0,t) &\leq \alpha_0 \int_{\partial\mathcal{B}_0 \cap B^3_{ct}(Z_0)} \left\{ \sum_{j_0=1}^{3} \{L_{Z_0}^{i_0 j_0}(Z_1)\}^2 \right\}^{1/2} \\ &\quad \cdot \{ct - r_{Z_0}(Z_1)\}\, d\lambda_{\partial\mathcal{B}_0}(Z_1) \\ &\leq (\alpha_0)^2 \int_0^{ct} (ct-s)\, ds \\ &= \frac{1}{2}(\alpha_0 ct)^2 \\ &\leq \frac{1}{2}(\alpha_0 cT)^2, \end{aligned} \tag{27}$$

while if $n \geq 3$, (II.5.2) is to be reapplied, showing that the innermost two integrations in (26) are bounded above by

$$\alpha_0 \int\limits_{\partial B_0 \cap B^3_{ct-\sum_{j=0}^{n-3} r_{Z_j}(Z_{j+1})}(Z_{n-2})} \left\{ \sum_{i_{n-2}=1}^{3} \sum_{j_{n-2}=1}^{3} (L_{Z_{n-2}}^{i_{n-2}j_{n-2}}(Z_{n-1}))^2 \right\}^{1/2}$$

$$\cdot \left\{ ct - \sum_{j=0}^{n-3} r_{Z_j}(Z_{j+1}) - r_{Z_{n-2}}(Z_{n-1}) \right\} d\lambda_{\partial B_0}(Z_{n-1})$$

$$\leq (\alpha_0)^2 \int\limits_0^{ct-\sum_{j=0}^{n-3} r_{Z_j}(Z_j+1)} \left\{ ct - \sum_{j=0}^{n-3} r_{Z_j}(Z_{j+1}) - s \right\} ds$$

$$= \frac{1}{2}(\alpha_0)^2 \cdot \left\{ ct - \sum_{j=0}^{n-3} r_{Z_j}(Z_{j+1}) \right\}^2 .$$

In fact, for $n \geq 3$ and $k \in \{1,\ldots,n-1\}$, one can prove by induction and (II.5.2) that the k innermost integrations in (26) are majorized by

$$\frac{1}{k!} (\alpha_0)^k \cdot \left\{ ct - \sum_{j=0}^{n-k-1} r_{Z_j}(Z_{j+1}) \right\}^k .$$

Finally, then, taking $k = n-1$ in the latter and using (II.5.1) to estimate the remaining integral in (26), we obtain

$$I_n^{i_0}(Z_0,t) \leq \frac{(\alpha_0)^{n-1}}{(n-1)!} \cdot \int\limits_{\partial B_0 \cap B^3_{ct}(Z_0)} \left\{ \sum_{j_0=1}^{3} (L_{Z_0}^{i_0 j_0}(Z_1))^2 \right\}^{1/2}$$

$$\cdot \{ct - r_{Z_0}(Z_1)\}^{n-1} d\lambda_{\partial B_0}(Z_1)$$

$$\leq \frac{(\alpha_0)^n}{(n-1)!} \cdot \int_0^{ct} (ct-s)^{n-1}\, ds \tag{28}$$

$$= \frac{1}{n!}(\alpha_0 ct)^n, \qquad \text{for} \qquad n \geq 3.$$

From (25), (27), and (28),

$$I_n^{i_0}(Z_0,t) \leq \frac{1}{n!}(\alpha_0 cT)^n \quad \text{whenever} \quad T > 0, \quad Z_0 \in \partial B_0, \quad t \in (0,T], \quad \text{and} \quad n \in \mathbb{N}. \tag{29}$$

Coupling (23) and (29), and agreeing to the convention

$$0^0 := 1, \tag{30}$$

we arrive at the inequality

$$\sum_{n=0}^{\infty} |\lambda^n \cdot \{\mathbb{L}^n f\}^i(Z,t)|$$

$$\leq b_{f,T} \cdot \sum_{n=0}^{\infty} \left\{\frac{1}{2\pi}(1+C_{f,T}T)\cdot|\lambda|\alpha_0 cT\right\}^n \cdot \frac{n^{\delta_{f,T}n}}{n!}, \tag{31}$$

valid for $T > 0$, $Z \in \partial B_0$, and $t \in [0,T]$.

Now, it is routine to verify that

$$\sum_{n=0}^{\infty} \frac{M^n}{n!} n^{\delta n} < \infty \quad \text{whenever} \quad M > 0 \quad \text{and} \quad \delta \in (0,1), \tag{32}$$

whence (31) implies that $\sum_{n=0}^{\infty} \lambda^n\{\mathbb{L}^n f\}^i$ converges absolutely on $\partial B_0 \times \mathbb{R}$ and uniformly on each compact subset of $\partial B_0 \times \mathbb{R}$; observe that

$\mathbb{L}^n f$ vanishes on $\partial\mathcal{B}_0\times(-\infty,0]$, for each n. Thus, we may define

$$\psi := \sum_{n=0}^{\infty} \lambda^n \mathbb{L}^n f,$$

obtaining a function $\psi \in C(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$ which vanishes on $\partial\mathcal{B}_0\times(-\infty,0]$, the continuity of ψ following from the fact that $\mathbb{L}^n f \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3) \subset C(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$ for each $n \in \mathbb{N}\cup\{0\}$ (cf., (II.3.30)) and the uniform convergence of the series on each compact subset of $\partial\mathcal{B}_0\times\mathbb{R}$. Note that, with (31), (4) certainly holds if $j = 0$ therein, in view of the definition (5).

We shall next show that $\psi \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$ and that the statements of (i.b) are correct (for $j \in \mathbb{N}$). Fix $p \in \mathbb{N}$, and consider the formal series of p^{th} 4-derivatives, $\sum_{n=0}^{\infty} \lambda^n D_4^p \mathbb{L}^n f$, or $\sum_{n=0}^{\infty} \lambda^n \mathbb{L}^n D_4^p f$ (cf., (II.3.28)). If $n \in \mathbb{N}$, $Z_0 \in \partial\mathcal{B}_0$, and $t > 0$, inequality (18) is valid when f is replaced therein by $D_4^p f$. If $n \in \mathbb{N}$ with $n \geq 2$, $T > 0$, $t \in (0,T]$, and $Z_0,\ldots,Z_n \in \partial\mathcal{B}_0$ with $t-\sum_{k=0}^{n-1} r_{Z_k}^c(Z_{k+1}) > 0$, we may follow the reasoning employed in (22), to obtain

$$\left|\left\{\left(\prod_{j=0}^{n-1} \{1+r_{Z_j}^c(Z_{j+1})D_4\}\right)D_4^p f\right\}\left(Z_n, t-\sum_{k=0}^{n-1} r_{Z_k}^c(Z_{k+1})\right)\right|_3$$

$$\leq \sum_{j=0}^{n} \alpha_j^{n-1}(r_{Z_\ell}^c(Z_{\ell+1}))\cdot\left|D_4^{p+j} f\left(Z_n, t-\sum_{k=0}^{n-1} r_{Z_k}^c(Z_{k+1})\right)\right|_3$$

$$\leq b_{f,T}\cdot C_{f,T}^p\cdot p^{(1+\delta_{f,T})p}$$

$$+ \sum_{j=1}^{n} \binom{n}{j}\cdot\left(\frac{t}{j}\right)^j\cdot b_{f,T}\cdot C_{f,T}^{p+j}\cdot(p+j)^{(1+\delta_{f,T})(p+j)}$$

$$\leq b_{f,T}\cdot C_{f,T}^{p}\Bigg\{p^{(1+\delta_{f,T})p}$$

$$+\sum_{j=1}^{n}\binom{n}{j}\cdot\left(\frac{1}{j}\right)^{j}\cdot(p+j)^{(1+\delta_{f,T})(p+j)}\cdot(C_{f,T}T)^{j}\Bigg\}\;. \tag{33}$$

One can check that the final inequality in (33) is also valid when $n = 1$. Thus, replacing f in (18) by $D_4^p f$, accounting for (33) and (29), and introducing the notational convenience

$$\left(\frac{1}{0}\right)^{0} := 1, \tag{34}$$

we are led to the inequality

$$\begin{aligned}
&|\{\mathbb{L}^n D_4^p f\}^{i_0}(Z_0,t)| \\
&\leq \left(\frac{1}{2\pi}\right)^{n}\cdot b_{f,T}C_{f,T}^{p}\left\{\sum_{j=0}^{n}\binom{n}{j}\cdot\left(\frac{1}{j}\right)^{j}\cdot(p+j)^{(1+\delta_{f,T})(p+j)}(C_{f,T}T)^{j}\right\}I_n^{i_0}(Z_0,t) \\
&\leq b_{f,T}\cdot C_{f,T}^{p}\left\{\sum_{j=0}^{n}\binom{n}{j}\cdot\left(\frac{1}{j}\right)^{j}\cdot(p+j)^{(1+\delta_{f,T})(p+j)}(C_{f,T}T)^{j}\right\}\cdot\frac{1}{n!}\cdot\left(\frac{\alpha_0 cT}{2\pi}\right)^{n},
\end{aligned} \tag{35}$$

valid for $n,p \in \mathbb{N}$, $T > 0$, $Z_0 \in \partial\mathcal{B}_0$, and $t \in [0,T]$.

Now, in Appendix II.A, it is proven that

$$\left(\frac{a+b}{2}\right)^{a+b} \leq a^a b^b \quad \text{whenever } a \text{ and } b \text{ are positive,}$$

whence it follows that, for any $\alpha > 0$,

$$(a+b)^{\alpha(a+b)} \leq 2^{\alpha(a+b)}\cdot a^{\alpha a}\cdot b^{\alpha b}.$$

Therefore, if $n,p \in \mathbb{N}$, $j \in \{1,\ldots,n\}$, and $\delta > 0$,

$$
\begin{aligned}
\left(\frac{1}{j}\right)^{j} \cdot (j+p)^{(1+\delta)(j+p)} &\le \left(\frac{1}{j}\right)^{j} \cdot 2^{(1+\delta)(j+p)} \cdot j^{(1+\delta)j} \cdot p^{(1+\delta)p} \\
&= 2^{(1+\delta)(j+p)} \cdot p^{(1+\delta)p} \cdot j^{\delta j} \qquad (36) \\
&\le 2^{(1+\delta)(j+p)} \cdot p^{(1+\delta)p} \cdot n^{\delta n}.
\end{aligned}
$$

Using (36) to continue the estimate begun in (35),

$$
\begin{aligned}
&|\{\mathbb{L}^{n} D_4^{p} f\}^{i_0}(Z_0,t)| \\
&\le b_{f,T} C_{f,T}^{p} \cdot \left\{1 + \sum_{j=1}^{n} \binom{n}{j} \cdot 2^{(1+\delta_{f,T})(p+j)} \cdot p^{(1+\delta_{f,T})p} \cdot n^{\delta_{f,T} n} \cdot (C_{f,T} T)^{j}\right\} \\
&\quad \cdot \frac{1}{n!} \cdot \left(\frac{\alpha_0 cT}{2\pi}\right)^{n} \\
&\le b_{f,T} \cdot \left\{2^{(1+\delta_{f,T})} C_{f,T}\right\}^{p} \cdot p^{(1+\delta_{f,T})p} \cdot \left\{\sum_{j=0}^{n} \binom{n}{j} \cdot \left\{2^{(1+\delta_{f,T})} C_{f,T} T\right\}^{j}\right\} \qquad (37) \\
&\quad \cdot \frac{n^{\delta_{f,T} n}}{n!} \cdot \left(\frac{\alpha_0 cT}{2\pi}\right)^{n} \\
&= b_{f,T} \cdot \left\{2^{(1+\delta_{f,T})} C_{f,T}\right\}^{p} \cdot p^{(1+\delta_{f,T})p} \cdot \left\{\frac{1}{2\pi} (1+2^{(1+\delta_{f,T})} C_{f,T} T) \cdot \alpha_0 cT\right\}^{n} \\
&\quad \cdot \frac{n^{\delta_{f,T} n}}{n!} ,
\end{aligned}
$$

holding for p, n, T, Z_0, and t as in (35).

It is evident that (37) is also true when $n = 0$; recall (30).

Consequently,

$$\sum_{n=0}^{\infty} |\lambda^n \cdot \{\mathbb{L}^n D_4^p f\}^i(Z,t)|$$

$$\leq b_{f,T} \cdot \left\{2^{(1+\delta_{f,T})} C_{f,T}\right\}^p \cdot p^{(1+\delta_{f,T})p}$$

$$\cdot \sum_{n=0}^{\infty} \left\{\frac{1}{2\pi} (1+2^{(1+\delta_{f,T})} C_{f,T} T) \cdot |\lambda| \alpha_0 cT\right\}^n \cdot \frac{n^{\delta_{f,T} n}}{n!}, \tag{38}$$

whenever $p \in \mathbb{N}$, $T > 0$, $Z \in \partial B_0$, and $t \in [0,T]$.

Taking note of (32), the estimates in (38) allow us to assert that, for each $p \in \mathbb{N}$, $\sum_{n=0}^{\infty} \lambda^n D_4^p \mathbb{L}^n f = \sum_{n=0}^{\infty} \lambda^n \mathbb{L}^n D_4^p f$ converges absolutely on $\partial B_0 \times \mathbb{R}$ and uniformly on each compact subset of $\partial B_0 \times \mathbb{R}$; in turn, again for each $p \in \mathbb{N}$, this implies that $D_4^p \psi$ exists and is continuous on $\partial B_0 \times \mathbb{R}$, with

$$D_4^p \psi = \sum_{n=0}^{\infty} \lambda^n \mathbb{L}^n D_4^p f. \tag{39}$$

Thus, $\psi \in C_4^{\infty}(\partial B_0 \times \mathbb{R}; \mathbb{K}^3)$. Directly from (38), we obtain (4) for $j \in \mathbb{N}$ (having already proven (4) for $j = 0$). Since we have pointed out that ψ vanishes on $\partial B_0 \times (-\infty, 0]$, inequalities (4) show that $\psi \in \mathcal{E}_{4,0}(\partial B_0 \times \mathbb{R}; \mathbb{K}^3)$.

We have now proven (i.b) ((8) is obvious), and need only verify (1) in order to complete the proof of (i.a). For this, note that we now know that the sequence of partial sums of the series $\sum_{k=0}^{\infty} \lambda^k \mathbb{L}^k f$, $(\psi_n = \sum_{k=0}^{n} \lambda^k \mathbb{L}^k f)_{n=0}^{\infty}$, converges uniformly on each compact subset of $\partial B_0 \times \mathbb{R}$ to ψ, while the sequence $(\psi_{n,4})_{n=0}^{\infty}$ possesses the same convergence characteristics and converges to

$\psi_{,4}$. Choose $Z \in \partial\mathcal{B}_0$ and $\zeta \in \mathbb{R}$. Then

$$[\psi_n]_{(Z,\zeta)}(Y) = \psi_n(Y,\zeta - r_Z^c(Y))$$

and

$$[\psi_{n,4}]_{(Z,\zeta)}(Y) = \psi_{n,4}(Y,\zeta - r_Z^c(Y))$$

while $(Y,\zeta - r_Z^c(Y))$ lies in the (compact) set $\partial\mathcal{B}_0 \times [\zeta - \frac{1}{c}\,\mathrm{diam}\,\mathcal{B}_0,\ \zeta]$ for each $Y \in \partial\mathcal{B}_0$. Therefore, $([\psi_n]_{(Z,\zeta)})_{n=0}^{\infty}$ and $([\psi_{n,4}]_{(Z,\zeta)})_{n=0}^{\infty}$ converge uniformly on $\partial\mathcal{B}_0$ to $[\psi]_{(Z,\zeta)}$ and $[\psi_{,4}]_{(Z,\zeta)}$, respectively, whence it is clear that

$$\lim_{n\to\infty} \{\mathbb{L}\psi_n\}^i(Z,\zeta)$$

$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial\mathcal{B}_0} L_Z^{ij}\cdot\{[\psi_n^j]_{(Z,\zeta)} + r_Z^c[\psi_{n,4}^j]_{(Z,\zeta)}\}\, d\lambda_{\partial\mathcal{B}_0}$$

$$= \frac{1}{2\pi} \int_{\partial\mathcal{B}_0} L_Z^{ij}\cdot\{[\psi^j]_{(Z,\zeta)} + r_Z^c[\psi^j_{,4}]_{(Z,\zeta)}\}\, d\lambda_{\partial\mathcal{B}_0}$$

$$= \{\mathbb{L}\psi\}^i(Z,\zeta),$$

so that

$$\lambda\mathbb{L}\psi = \lim_{n\to\infty} \lambda\mathbb{L}\left(\sum_{k=0}^{n} \lambda^k\mathbb{L}^k f\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \lambda^k\mathbb{L}^k f \tag{40}$$

$$= \sum_{k=1}^{\infty} \lambda^k\mathbb{L}^k f \quad \text{on} \quad \partial\mathcal{B}_0\times\mathbb{R}.$$

Immediately from (40), we produce the desired equality

$$f+\lambda\mathbb{L}\psi = f+\sum_{k=1}^{\infty} \lambda^k\mathbb{L}^k f = \sum_{k=0}^{\infty} \lambda^k\mathbb{L}^k f = \psi \quad \text{on} \quad \partial\mathcal{B}_0\times\mathbb{R},$$

which is just (1).

Finally, to prove (i.c), suppose that $f \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$. We have already seen that $\psi \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$, so (II.3.30) gives $\mathbb{L}\psi \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3)$, with which (1) shows that

$$\psi = f+\lambda\mathbb{L}\psi \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{K}^3).$$

The proof of (i) is now complete.

(ii) The proof of this second half of the theorem parallels that of the first so closely that we shall but touch upon the major steps. We begin by defining the successive approximations $\{\Psi_n\}_{n=0}^{\infty}$ according to

$$\Psi_0 := F,$$

$$\Psi_n := F+\lambda L\Psi_{n-1} \quad \text{for each} \quad n \in \mathbb{N},$$

discover that

$$\Psi_n = \sum_{k=0}^{n} \lambda^k L^k F \quad \text{for each} \quad n \in \mathbb{N}\cup\{0\},$$

and so are motivated to examine the formal series $\sum_{k=0}^{\infty} \lambda^k L^k F$.

Starting from (II.3.24) and proceeding essentially as in the derivation of (23) (of course, Cauchy's inequality is not needed), one can show that

$$|L^n F(Z_0,t)| \leq b_{F,T} \cdot \left\{\frac{1}{2\pi}\,(1+C_{F,T}T)\right\}^n \cdot n^{\delta_{F,T}n} \cdot \tilde{I}_n(Z_0,t) \tag{41}$$

for $T > 0$, $Z_0 \in \partial\mathcal{B}_0$, $t \in (0,T]$, and $n \in \mathbb{N}$,

with

$$\tilde{I}_n(Z_0,t) := \int_{\partial\mathcal{B}_0(Z_0;t)} \cdots \int_{\partial\mathcal{B}_0(Z_0,\ldots,Z_{n-1};t)} \prod_{k=0}^{n-1} |L_{Z_k}(Z_{k+1})|\; d\lambda_{\partial\mathcal{B}_0}(Z_n) \ldots d\lambda_{\partial\mathcal{B}_0}(Z_1). \tag{42}$$

Appealing to Lemma [II.5], in particular (II.5.3), the companion to (29) can be easily secured:

$$\tilde{I}_n(Z_0,t) \leq \frac{1}{n!}\,(\alpha_0 cT)^n \quad \textit{whenever} \quad T > 0, \quad Z_0 \in \partial\mathcal{B}_0, \quad t \in (0,T], \quad \text{and} \quad n \in \mathbb{N}. \tag{43}$$

Thus,

$$\sum_{n=0}^{\infty} |\lambda^n \cdot L^n F(Z_0,t)| \leq b_{F,T} \cdot \sum_{n=0}^{\infty} \left\{\frac{1}{2\pi}\,(1+C_{F,T}T)\cdot|\lambda|\alpha_0 cT\right\}^n \cdot \frac{n^{\delta_{F,T}n}}{n!} \tag{44}$$

for $T > 0$, $Z_0 \in \partial\mathcal{B}_0$, and $t \in [0,T]$.

With (32), we can conclude that the series in question converges absolutely and uniformly on each compact subset of $\partial\mathcal{B}_0 \times \mathbb{R}$, so that we may define

$$\Psi := \sum_{n=0}^{\infty} \lambda^n L^n F, \quad \text{on} \quad \partial\mathcal{B}_0\times\mathbb{R};$$

Ψ is continuous and vanishes on $\partial\mathcal{B}_0\times(-\infty,0]$. (44) implies (11) when $j = 0$. The inequalities

$$\sum_{n=0}^{\infty} |\lambda^n\cdot L^n D_4^p F(Z,t)|$$

$$\le b_{F,T}\cdot\left\{2^{(1+\delta_{F,T})} C_{F,T}\right\}^p\cdot p^{(1+\delta_{F,T})p}$$

$$\cdot \sum_{n=0}^{\infty} \left\{\frac{1}{2\pi}\,(1+2^{(1+\delta_{F,T})} C_{F,T}T)\cdot|\lambda|\alpha_0 cT\right\}^n\cdot \frac{n^{\delta_{F,T}n}}{n!}, \tag{45}$$

valid for $p \in \mathbb{N}$, $T > 0$, $Z \in \partial\mathcal{B}_0$, and $t \in (0,T]$,

can be deduced by following the arguments which led to (38). Consequently, if $p \in \mathbb{N}$, $\sum_{n=0}^{\infty} \lambda^n D_4^p L^n F = \sum_{n=0}^{\infty} \lambda^n L^n D_4^p F$ converges absolutely and uniformly on each compact subset of $\partial\mathcal{B}_0\times\mathbb{R}$. Thus, $\Psi \in C_4^{\infty}(\partial\mathcal{B}_0\times\mathbb{R})$, with

$$D_4^p\Psi = \sum_{n=0}^{\infty} \lambda^n L^n D_4^p F \quad \text{for each} \quad p \in \mathbb{N}.$$

From (45), it is now seen that the remaining estimates in (11) hold, whence $\Psi \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$. Equality (9) is a result of the uniform convergence on compact subsets of $\partial\mathcal{B}_0\times\mathbb{R}$ of the series for Ψ and $\Psi_{,4}$, and can be checked by retracing the proof of (1), *mutatis mutandis*. Finally, (ii.c) is an obvious consequence of (9), the inclusion $\Psi \in \mathcal{E}_{4,0}(\partial\mathcal{B}_0\times\mathbb{R})$, and the mapping property of L given by (II.3.29). $\square$.

[II.8] R E M A R K. If, in [II.7], $\mathcal{B}_0$ does not fulfill condition (G), then the reasoning of the proof can still be used to prove that there exist solutions of the equations considered on $\partial\mathcal{B}_0\times(-\infty,d]$, if $(a,1,d)$ is a set of Lyapunov constants for $\mathcal{B}_0$. To provide the wherewithal for continuing this solution, we should return and develop a local existence theorem for solutions of the equations in [II.7] which satisfy more general initial conditions on an appropriate set $\partial\mathcal{B}_0\times[\gamma,0]$, $\gamma < 0$.

With the aid of Theorem [II.7], we can show that the reformulation work of Chapter 6 in Part I leads to an existence result for a certain class of scattering problems in the case of a stationary body.

[II.9] T H E O R E M. *Let* M *be a null motion in* $\mathbb{M}(2)$, *and assume that* $\mathcal{B}_0$ *satisfies condition* (G). *Let* $\{E^{ii},B^{ii}\}$ *be an incident field appropriate to* M *as in* [I.4.1], *for which it is also known that* F_1 *and* F_{-1} *are in* $\mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R})$, *while* f_1 *and* f_{-1} *lie in* $\mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R}^3)$, *wherein* F_1, F_{-1}, f_1, *and* f_{-1} *are given on* $\partial\mathcal{B}_0\times\mathbb{R}$ *by* (II.4.2-5), *respectively. Then there exists a (unique) solution to the scattering problem generated by* M *and* $\{E^{ii},B^{ii}\}$. *This solution is given by either*

$$E^{\sigma i^c}(X,t) = -V^0\{\Psi_1\}_{,i}(X,t) - \frac{1}{c}\,V^0\{\psi_1^i\}_{,4}(X,t)$$

$$= \frac{1}{4\pi}\int_{\partial\mathcal{B}_0}\left(\frac{1}{r_X}\right)_{,i}\cdot[\Psi_1](X,t)\;d\,\partial\mathcal{B}_0$$

$$- \frac{1}{4\pi c} \int_{\partial B_0} \frac{1}{r_X} r_{X,i} \cdot [\Psi_{1,4}](X,t) \; d\lambda_{\partial B_0} \tag{1}$$

$$- \frac{1}{4\pi c} \int_{\partial B_0} \frac{1}{r_X} \cdot [\psi^i_{1,4}](X,t) \; d\lambda_{\partial B_0},$$

$$\begin{aligned} B^{\sigma i}(X,t) &= \varepsilon_{ijk} V^0\{\psi^k_1\}_{,j}(X,t) \\ &= - \frac{1}{4\pi} \int_{\partial B_0} \varepsilon_{ijk} \left(\frac{1}{r_X}\right)_{,j} \cdot [\psi^k_1](X,t) \; d\lambda_{\partial B_0} \\ &\quad + \frac{1}{4\pi c} \int_{\partial B_0} \frac{1}{r_X} \varepsilon_{ijk} r_{X,j} \cdot [\psi^k_{1,4}](X,t) \; d\lambda_{\partial B_0}, \end{aligned} \tag{2}$$

for each $X \in B_0'$, $t \in \mathbb{R}$,

with $\psi_1 \in \mathcal{E}^H_{4,0}(\partial B_0 \times \mathbb{R}; \mathbb{R}^3)$ *and* $\Psi_1 \in \mathcal{E}^H_{4,0}(\partial B_0 \times \mathbb{R}; \mathbb{R})$ *to be obtained from*

$$\psi_1 := \sum_{n=0}^{\infty} \mathbb{L}^n f_1 = 2 \sum_{n=0}^{\infty} \mathbb{L}^n\{\nu \times (B^i | \; \partial B_0 \times \mathbb{R})\} \tag{3}$$

and

$$\Psi_1 := \sum_{n=0}^{\infty} L^n\{F_1 + \Lambda\psi_1\} = \sum_{n=0}^{\infty} L^n\{2\nu \bullet (E^{ic} | \; \partial B_0 \times \mathbb{R}) + \Lambda\psi_1\}, \tag{4}$$

or

$$\begin{aligned} E^{\sigma i^c}(X,t) &= -\varepsilon_{ijk} V^0\{\psi^k_{-1}\}_{,j}(X,t) \\ &= \frac{1}{4\pi} \int_{\partial B_0} \varepsilon_{ijk} \left(\frac{1}{r_X}\right)_{,j} \cdot [\psi^k_{-1}](X,t) \; d\lambda_{\partial B_0} \end{aligned}$$

$$- \frac{1}{4\pi c} \int_{\partial B_0} \frac{1}{r_X} \varepsilon_{ijk} r_{X,j} \cdot [\psi^k_{-1,4}](X,t)\ d\lambda_{\partial B_0}, \tag{5}$$

$$\begin{aligned} B^{\sigma i}(X,t) &= -V^0\{\Psi_{-1}\}_{,i}(X,t) - \frac{1}{c} V^0\{\psi^i_{-1}\}_{,4}(X,t) \\ &= \frac{1}{4\pi} \int_{\partial B_0} \left(\frac{1}{r_X}\right)_{,i} \cdot [\Psi_{-1}](X,t)\ d\lambda_{\partial B_0} \\ &\quad - \frac{1}{4\pi c} \int_{\partial B_0} \frac{1}{r_X} r_{X,i} \cdot [\Psi_{-1,4}](X,t)\ d\lambda_{\partial B_0} \\ &\quad - \frac{1}{4\pi c} \int_{\partial B_0} \frac{1}{r_X} \cdot [\psi^i_{-1,4}](X,t)\ d\lambda_{\partial B_0}, \end{aligned} \tag{6}$$

for each $X \in B_0'$, $t \in \mathbb{R}$,

wherein $\psi_{-1} \in \mathcal{E}^H_{4,0}(\partial B_0 \times \mathbb{R}; \mathbb{R}^3)$ *and* $\Psi_{-1} \in \mathcal{E}^H_{4,0}(\partial B_0 \times \mathbb{R}; \mathbb{R})$ *are defined by*

$$\begin{aligned} \psi_{-1} &:= \sum_{n=0}^{\infty} (-1)^{n+1} \mathbb{L}^n f_{-1} \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \mathbb{L}^n \{\nu \times (E^1 | \ \partial B_0 \times \mathbb{R})\}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \Psi_{-1} &:= \sum_{n=0}^{\infty} (-1)^{n+1} L^n \{F_{-1} + \Lambda\psi_{-1}\} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} L^n \{2\nu \bullet (B^1 | \ \partial B_0 \times \mathbb{R}) + \Lambda\psi_{-1}\}; \end{aligned} \tag{8}$$

the series appearing in (3), (4), (7), *and* (8) *converge absolutely*

and uniformly on each compact subset of $\partial B_0 \times \mathbb{R}$. *Moreover,*

$$E^{\sigma i} \text{ and } B^{\sigma i} \in C^{\infty}(B_0' \times \mathbb{R};\mathbb{R}) \cap C(B_0'^{-} \times \mathbb{R};\mathbb{R}). \tag{9}$$

All partial derivatives of $E^{\sigma i}$ *and* $B^{\sigma i}$ *can be computed from either* (1) *and* (2) *or* (5) *and* (6), *respectively, by differentiation under the integrals appearing; the 4-derivatives of* ψ_1 *and* Ψ_1 *or* ψ_{-1} *and* Ψ_{-1} *which occur thereby can be computed via term-by-term differentiation of the defining series* (3) *and* (4) *or* (7) *and* (8), *respectively, each differentiated series converging absolutely and uniformly on each compact subset of* $\partial B_0 \times \mathbb{R}$. *Estimates for* ψ_1, Ψ_1, ψ_{-1}, *and* Ψ_{-1} *and their 4-derivatives, hence also for* $E^{\sigma i}$ *and* $B^{\sigma i}$ *and their partial derivatives, can be derived by applying the results of* [II.7]. *Further relations amongst* ψ_1, Ψ_1, ψ_{-1}, Ψ_{-1}, $E^{\sigma}| \, \partial B_0 \times \mathbb{R}$, *and* $B^{\sigma}| \, \partial B_0 \times \mathbb{R}$ *are contained in the conclusions of* [I.6.1].

Before proving these statements, we point out that if the restrictions $E^{\iota}| \, \partial B_0 \times \mathbb{R}$ and $B^{\iota}| \, \partial B_0 \times \mathbb{R}$ are known to lie in $\mathcal{E}^H_{4,0}(\partial B_0 \times \mathbb{R};\mathbb{R}^3)$, then certainly the conditions required here of f_1, F_1, f_{-1}, and F_{-1} are fulfilled. For example, if E^{ι} and B^{ι} are in $C^{\infty}(\Omega^{\iota};\mathbb{R}^3)$, with $\{D_4^j E^{\iota}| \, \partial B_0 \times \mathbb{R}\}_{j=0}^{\infty}$ and $\{D_4^j B^{\iota}| \, \partial B_0 \times \mathbb{R}\}_{j=0}^{\infty}$ satisfying the estimates of (II.2.2), then $E^{\iota}| \, \partial B_0 \times \mathbb{R}$ and $B^{\iota}| \, \partial B_0 \times \mathbb{R}$ are in $C^2(\partial B_0 \times \mathbb{R};\mathbb{R}^3)$, whence they are locally Lipschitz continuous, and so belong to $\mathcal{E}^H_{4,0}(\partial B_0 \times \mathbb{R};\mathbb{R}^3)$.

P R O O F. According to [I.6.1] and [I.6.5], we can show that the

scattering problem corresponding to $\mathcal{M}$ and $\{E^{1i}, B^{1i}\}$ possesses a solution if we can solve the (modified) reformulated problem: show that there exist locally Hölder continuous functions ψ_1, Ψ_1, ψ_{-1}, and Ψ_{-1} on $\partial\mathcal{B}_0\times\mathbb{R}$, vanishing on $\partial\mathcal{B}_0\times(-\infty,0]$, with

$$\left.\begin{aligned} &D_4^j\psi_1,\ D_4^j\psi_{-1} \in C(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R}^3),\\ &D_4^j\Psi_1,\ D_4^j\Psi_{-1} \in C(\partial\mathcal{B}_0\times\mathbb{R}),\end{aligned}\right\}\quad \text{for}\quad j = 1 \text{ and } 2,$$

while ψ_1 and Ψ_1 are solutions of (I.6.5.4), and ψ_{-1} and Ψ_{-1} comprise a solution of (I.6.5.6), *i.e.*, in view of the results of [II.1] and [II.4], such that

$$\left.\begin{aligned} \Psi_\lambda-\lambda L\Psi_\lambda &= \lambda\cdot F_\lambda+\lambda\cdot\Lambda\psi_\lambda, \qquad (10)_1\\ \psi_\lambda-\lambda\mathbb{L}\psi_\lambda &= \lambda\cdot f_\lambda, \qquad (10)_2\end{aligned}\right\}\quad \text{on}\quad \partial\mathcal{B}_0\times\mathbb{R},\quad \text{for}\quad \lambda = 1 \text{ and } -1.$$

Once the existence of such functions has been established, a solution of the scattering problem can be constructed by using either (I.6.1.6 and 7) (with $\psi^i = \psi_1^i$ and $\Psi = \Psi_1$ therein) or (I.6.1.8 and 9) (with $\gamma^i = \psi_{-1}^i$ and $\Gamma = \Psi_{-1}$ therein); in fact, all conclusions of [I.6.1] will be valid (with the appropriate replacements of symbols).

Now, with $\lambda = 1$ $[\lambda = -1]$, since $f_\lambda \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R}^3)$, [II.7.i] asserts that $(10)_2$ holds when ψ_λ is given by (3) [by (7)], that $\psi_\lambda \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R}^3)$, and that the series (3) [(7)] as well as those giving $\{D_4^j\psi_\lambda\}_{j=1}^\infty$ possess the convergence properties claimed for them. Next, (II.3.31) implies that $\Lambda\psi_\lambda \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R})$,

so $F_\lambda + A\psi_\lambda \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R})$. Thus, we may apply [II.7.ii] to conclude that $(10)_1$ obtains if Ψ_λ is defined by (4) [by (8)], that $\Psi_\lambda \in \mathcal{E}^H_{4,0}(\partial\mathcal{B}_0\times\mathbb{R};\mathbb{R})$, and that the series (4) [(8)] as well as those giving $\{D_4^j\Psi_\lambda\}_{j=1}^\infty$ have the convergence characteristics claimed for them.

Since we have produced for the reformulated problem a solution of the required form, we know that the scattering problem induced by the data M and $\{E^{\iota i}, B^{\iota i}\}$ is also solvable, a solution being given by either (from (I.6.1.6 and 7))

$$E^{\sigma i^c} = -V^0\{\Psi_1\}_{,i} - \frac{1}{c}\, V^0\{\psi_1^i\}_{,4}, \tag{11}$$

$$B^{\sigma i} = \varepsilon_{ijk} V^0\{\psi_1^k\}_{,j} \tag{12}$$

or (from (I.6.1.8 and 9))

$$E^{\sigma i^c} = -\varepsilon_{ijk} V^0\{\psi_{-1}^k\}_{,j}, \tag{13}$$

$$B^{\sigma i} = -V^0\{\Psi_{-1}\}_{,i} - \frac{1}{c}\, V^0\{\psi_{-1}^i\}_{,4}. \tag{14}$$

Now, explicit expressions for the partial derivatives of $V^0\{\psi_1^i\}$, $V^0\{\Psi_1\}$, $V^0\{\psi_{-1}^i\}$, and $V^0\{\Psi_{-1}\}$ are available from equalities (I.5.13.2 and 3); using these in (11)-(14), one can easily check that (1), (2), (5), and (6) are correct.

The inclusions $E^{\sigma i}$, $B^{\sigma i} \in C^1(\mathcal{B}_0'\times\mathbb{R};\mathbb{R})\cap C(\mathcal{B}_0'^-\times\mathbb{R};\mathbb{R})$ follow from [I.6.1]. But, since $M \in \mathbb{M}(2;\infty)$ and ψ_1^i, $\Psi_1 \in C_4^\infty(\partial\mathcal{B}_0\times\mathbb{R})$ (or ψ_{-1}^i, $\Psi_{-1} \in C_4^\infty(\partial\mathcal{B}_0\times\mathbb{R})$), it is clear from [I.5.7] and the representations (1) and (2) (or (5) and (6)) that $E^{\sigma i}$ and $B^{\sigma i}$ are in

$C^{\infty}(\mathcal{B}_0' \times \mathbb{R})$, while the partial derivatives of these functions can be computed by differentiation under the integrals in (1) and (2) (or (5) and (6)). As already noted, all 4-derivatives of ψ_1, Ψ_1, ψ_{-1}, and Ψ_{-1} can be computed by term-by-term differentiation of the respective defining series, as [II.7] shows.

Finally, the uniqueness of this solution of the scattering problem is an immediate consequence of [I.4.10]. □.

[II.10] R E M A R K S. (a) If, in [II.9], $\mathcal{B}_0$ does not satisfy condition (G), then we can still construct a solution of the scattering problem for $t \leq d$, wherein $(a,1,d)$ is a set of Lyapunov constants for $\mathcal{B}_0$: cf., Remark [II.8], In order to prove that this solution can be continued, we might proceed by either developing an existence theorem for the equations considered in [II.7] without imposing condition (G), or solving a Cauchy problem for Maxwell's equations and using the result to set up an auxiliary scattering problem with homogeneous initial conditions at $t = d$, solving this for $t \leq 2d$, *etc*. For construction purposes, the latter stepwise procedure would obviously be at best cumbersome.

II.A. APPENDIX

AN INEQUALITY

In the proof of Theorem [II.7], use is made of the inequality verified in the following statement.

L E M M A. *Let* a *and* b *be positive numbers. Then*

$$\left(\frac{a+b}{2}\right)^{a+b} \leq a^a b^b, \tag{1}$$

equality holding iff a = b.

P R O O F. Clearly, (1) is true iff

$$(a+b)\ \ln\left(\frac{a+b}{2}\right) \leq a\ \ln a + b\ \ln b,$$

or

$$(a+b)\ \ln\ (a+b) \leq a\ \ln a + b\ \ln b + (a+b)\ \ln 2: \tag{2}$$

we shall prove (2). Setting $\alpha := a/b$, we have

$$(a+b)\ \ln\ (a+b) = a\ \ln a + b\ \ln b + a\ \ln\left(1+\frac{b}{a}\right) + b\ \ln\left(1+\frac{a}{b}\right)$$

$$= a \ln a + b \ln b + (a+b) \cdot \left\{ \frac{\alpha}{1+\alpha} \ln \left(1 + \frac{1}{\alpha}\right) + \frac{1}{1+\alpha} \ln (1+\alpha) \right\} . \tag{3}$$

Thus, we are led to examine the function f given on $(0,\infty)$ by

$$f(x) := \frac{x}{1+x} \ln \left(1 + \frac{1}{x}\right) + \frac{1}{1+x} \ln (1+x)$$

$$= \frac{1}{1+x} \{(1+x) \ln (1+x) - x \ln x\}, \qquad \text{for} \quad x > 0.$$

We find

$$f'(x) = - \frac{\ln x}{(1+x)^2}, \qquad \text{for} \quad x > 0,$$

whence it is easy to see that f takes on its absolute maximum at the single point 1, where $f(1) = \ln 2$. Since (3) says that

$$(a+b) \ln (a+b) = a \ln a + b \ln b + (a+b) \cdot f(\alpha),$$

it follows that (2) is true, with equality holding iff $\alpha = 1$, *i.e.*, iff $a = b$. This completes the proof. □.

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†Observe that

$$a \ln \left(1 + \frac{b}{a}\right) + b \ln \left(1 + \frac{a}{b}\right) < a \cdot \frac{b}{a} + b \cdot \frac{a}{b} = (a+b) \ln e:$$

in view of the first equality in (3), this shows that (2), hence also (1), is true with strict inequality and 2 replaced by e. This actually suffices for the requirements of the proof of [II.7].

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